SETS OF INDEPENDENT POSTULATES FOR BETWEENNESS*

BY

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Introduction

The "universe of discourse" of the present paper is the class of all well-defined systems (K, R) where K is any class of elements $A, B, C, \cdots$, and R is any triadic relation. The notation $\mathrm{R}[ABC]$, or simply ABC, indicates that three given elements A, B, C, in the order stated, satisfy the relation R.

Examples of such systems (K, R) are the following, of which example (*a*) is the most important:

(*a*) K is the class of points on a line; AXB means that the point X lies between the points A and B.

(*b*) K is the class of natural numbers; AXB means that the number X is the product of the numbers A and B.

(*c*) K is the class of human beings; AXB means that X is a descendant of A and an ancestor of B.

(*d*) K is the class of points on the circumference of a circle; AXB means that the arc A-X-B is less than 180°.

(*e*) K is a class comprising four elements, namely, the numbers $2, 6, -6$, and 648; AXB means $X^4 = A \times B$.

It is obvious that these systems, and others like them, will possess a great variety of properties expressible in terms of the fundamental variables K and R. The object of this paper is to state clearly the characteristic properties of the type of system represented by example (*a*) above, by which this type of system is distinguished from all other possible systems (K, R).

In Section 1, we give a basic list of twelve postulates, due essentially to Pasch,† from which various sets of *independent* postulates will later be selected.

* Presented to the Society, September 5, 1916. The parts of the paper which do not involve the postulates here numbered 5 and 8 were presented by Professor Huntington at the meetings of December 31, 1912, and April 26, 1913. The necessity of adding postulates 5 and 8 was kindly pointed out by Professor R. L. Moore, and all the theorems and examples which involve these two postulates are due to Dr. Kline.

† M. Pasch, *Vorlesungen über neuere Geometrie*, Leipzig, 1882; G. Peano, *Sui fondamenti della geometria*, Rivista di Matematica, vol. 4 (1894), pp. 51–90; F. Schur, *Grundlagen der Geometrie*, Leipzig, 1909. Other sets of postulates for betweenness have been given

These postulates are all "general laws" as distinguished from "existence postulates," and include, in fact, all the possible general laws of linear order concerning not more than four elements.

In Sections 2 and 3 we give an exhaustive discussion of all the possible ways in which any one of these basic postulates can be deduced from any others of the list, and in Section 4, we give an exhaustive list of all the distinct sets of independent postulates (eleven in number) which can be selected from the basic list.*

Any one of these sets of independent postulates may be used, as in Section 5, to define the type of system (K, R) which we are considering—that is, to define the relation of betweenness.

The existence postulates which might be imposed, in addition to the general laws, would serve to distinguish the various sub-types which are included within the general type of system (K, R) here considered. These existence postulates, such as the postulates of discreteness, density, continuity, etc., are already well known, and will not be discussed further in the present paper.†

1. Basic list of twelve postulates

In this section we give the basic list of twelve postulates from which various sets of independent postulates will later be selected.

The first four postulates, A–D, concern three elements.

Postulate A. $AXB \,.\supset.\, BXA$.

That is, if AXB is true, then BXA is true. In other words, in the notation ABC, an interchange of the terminal elements is always allowable.

Postulate B.

$$A \neq B \,.\, B \neq C \,.\, C \neq A :\supset: BAC \smile CAB \smile ABC \smile CBA \smile ACB \smile BCA.$$

That is, if A, B, C, are distinct, then *at least one* of the three elements will occupy the middle position in a true triad.

Postulate C. $A \neq X \,.\, X \neq Y \,.\, Y \neq A :\supset: AXY \,.\, AYX \,.=.\, 0$.

by D. Hilbert; *Grundlagen der Geometrie*, 1899, third edition 1909; and by O. Veblen: *A system of axioms for geometry*, these Transactions, vol. 5 (1904), pp. 343–384, or *The foundations of geometry*, in the volume called *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young, 1911, pp. 1–51.

* The postulates of each of these sets are independent of each other in the ordinary sense of the term "independence"; that is, no postulate of any one set can be deduced from the remaining postulates of that set. It is probable that the postulates of each set are also "completely independent" in the sense suggested by E. H. Moore in his *Introduction to a Form of General Analysis* (*New Haven Colloquium*, 1906, published by the Yale University Press, New Haven, 1910, p. 82); but no attempt to discuss the "complete existential theory" of the postulates, in the sense there defined, has here been made.

† See, for example, E. V. Huntington, *The continuum as a type of order*, reprinted from the Annals of Mathematics, 1905 (Publication Office of Harvard University); second edition, Harvard University Press, 1917.

That is, if A, X, Y, are distinct, we cannot have AXY and AYX both true at the same time.

From Postulates A and C it follows that if A, B, C are distinct elements, then *not more than one* of the three elements can occupy the middle position in a true triad.

From Postulates A, B, and C, together, it follows that if A, B, and C are distinct elements, then one and only one of the triads ABC, BCA, CAB will be true.

Postulate D. $ABC : \supset : A \neq B \,.\, B \neq C \,.\, C \neq A$.

That is, if ABC is true, then the elements A, B, and C, are distinct.

The remaining eight postulates are concerned with four distinct elements.

Postulates 1–8. If

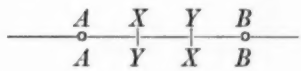
$$A \neq B \,.\, A \neq X \,.\, A \neq Y \,.\, B \neq X \,.\, B \neq Y \,.\, X \neq Y,$$

then

1. $XAB \,.\, ABY \,.\supset.\, XAY$;

2. $XAB \,.\, AYB \,.\supset.\, XAY$;
3. $XAB \,.\, AYB \,.\supset.\, XYB$;

4. $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$;
5. $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$;



6. $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$;
7. $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$;
8. $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.

The eight Postulates 1–8 (together with the analogous postulates obtained from these by the aid of Postulate A alone) include *all the possible "general laws" of betweenness concerning four distinct elements.* For, if we think of A and B as two given points on a line, the hypotheses of these postulates state all the possible relations in which two other distinct points X and Y of the line can stand in regard to A and B. (See, however, the Appendix.)

We shall see later that no further general laws—that is, no general laws concerning more than four distinct elements—need be assumed as fundamental. (Existence postulates, which play a very different rôle from the general laws, are not here considered.)

2. Theorems on deducibility

In this section we take up all the cases in which the following question is to be answered in the affirmative:

Given, any subset S of the twelve postulates of our basic list, and any postulate P of the list, not belonging to S; is P deducible from S?

The answers are comprised in the following 71 theorems. In the proofs of these theorems, all the steps are given explicitly, except those depending only on Postulate A. Moreover, in case any postulate (except Postulate A) is used more than once in a proof, the frequency of its use is indicated by an exponent; this latter information, however, is added merely as a matter of possible interest to the reader, and the omission of the exponents would not affect the conclusions of the paper in any way.*

A summary of the theorems will be found at the end of § 2.

Proofs of postulate 1

THEOREM 1a. *Proof of* 1 *from* A, B, C^3, 2, 4.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile AXY$. Suppose AXY. Then by 4, $ABY \,.\, AXY \,.\supset.\, ABX \smile AXB$, contrary to XAB, by C. Suppose AYX. Then by 2, $BAX \,.\, AYX \,.\supset.\, BAY$, contrary to ABY, by C. Therefore XAY.

THEOREM 1b. *Proof of* 1 *from* A, B, C^2, 3, 4.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By C, AXB and ABX are false, since XAB is true. By B, $XYA \smile AXY \smile XAY$.

Case 1. Suppose XYA. Then by 3, $XYA \,.\, YBA \,.\supset.\, XBA$, which is false.

Case 2. Suppose AXY. Then by 4, $AXY \,.\, ABY \,.\supset.\, AXB \smile ABX$, which are both false. Therefore XAY.

THEOREM 1c. *Proof of* 1 *from* A, B, C^3, 2^2, 5.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile AXY$.

Suppose AXY. Then by 5, $AXY \,.\, ABY \,.\supset.\, AXB \smile BXY$. But AXB is contrary to XAB, by C; while if BXY, then by 2, $ABY \,.\, BXY \,.\supset.\, ABX$, contrary to XAB, by C.

Suppose AYX. Then by 2, $BAX \,.\, AYX \,.\supset.\, BAY$, contrary to ABY, by C. Therefore XAY.

THEOREM 1d. *Proof of* 1 *from* A, B, C, 3^2, 5.

To prove: $XAB \,.\, ABY \,.\supset.\, XAY$. By C, XBA is false, since XAB is true. By B, $XYA \smile AXY \smile XAY$.

Case 1. Suppose XYA. Then by 3, $XYA \,.\, YBA \,.\supset.\, XBA$, which is false.

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* We are indebted to Mr. R. M. Foster, of Harvard University, for reductions in the "frequency exponents" (chiefly in regard to Postulate C) in the following theorems: 1b, 1d; 2c, 2g; 3a, 3b; 4b; 5f; 6j; 7b, 7c, 7j; 8b, 8c, 8d, 8f, 8j. A notion similar to that of "frequency exponents" was introduced by H. Brandes in his Halle Dissertation, 1908, *Über die axiomatische Einfachheit, mit besonderer Berücksichtigung der auf Addition beruhenden Zerlegungsbeweise des Pythagoräischen Lehrsatzes.* Compare F. Bernstein, *Ueber die axiomatische Einfachheit von Beweisen*, Atti del IV Congresso Internazionale dei Matematici, Roma, 1908, vol. 3 (1909), pp. 391–392, and also E. Lemoine's *Géométrographie* of 1902.

Case 2. If AXY, then by 5, $ABY \,.\, AXY \,.\supset.\, ABX \smile XBY$; but ABX is false, and if XBY, then by 3, $YBX \,.\, BAX \,.\supset.\, YAX$. Therefore XAY.

Proofs of postulate 2

Theorem 2a. *Proof of* 2 *from* $A, B, C^3, 1^3, 7$.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile AXY$. Suppose AYX. Then by 7, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBY$. But if BXY, then by 1, $BXY \,.\, XYA \,.\supset.\, BXA$, contrary to XAB, by C; and if XBY, then by 1, $XBY \,.\, BYA \,.\supset.\, XBA$, contrary to XAB, by C. Suppose AXY. Then by 1, $BAX \,.\, AXY \,.\supset.\, BAY$, contrary to AYB, by C. Therefore XAY.

Theorem 2b. *Proof of* 2 *from* $A, B, C^3, 1, 6$.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile YXA$. Suppose YXA. Then by 1, $BAX \,.\, AXY \,.\supset.\, BAY$, contrary to AYB, by C. Suppose AYX. Then by 6, $XYA \,.\, BYA \,.\supset.\, XBA \smile BXA$, contrary to XAB, by C. Therefore XAY.

Theorem 2c. *Proof of* 2 *from* $A, B, C^2, 3, 6$.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By C, BXA and XBA are false, since XAB is true. By B, $YXA \smile XYA \smile XAY$.

Case 1. Suppose YXA. Then by 3, $BYA \,.\, YXA \,.\supset.\, BXA$, which is false.

Case 2. Suppose XYA. Then by 6, $BYA \,.\, XYA \,.\supset.\, BXA \smile XBA$, which are both false. Therefore XAY.

Theorem 2d. *Proof of* 2 *from* $A, C, 3^2, 7$.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By 3, $XAB \,.\, AYB \,.\supset.\, XYB$. Hence by 7, $XYB \,.\, AYB \,.\supset.\, XAY \smile AXY$. But if AXY, then by 3, $BYA \,.\, YXA \,.\supset.\, BXA$, contrary to XAB, by C. Therefore XAY.

Theorem 2e. *Proof of* 2 *from* $A, C^2, 3, 4, 6$.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By 3, $XAB \,.\, AYB \,.\supset.\, XYB$. Hence by 4, $XYB \,.\, XAB \,.\supset.\, XYA \smile XAY$. But if XYA, then by 6, $XYA \,.\, BYA \,.\supset.\, XBA \smile BXA$, contrary to XAB, by C. Therefore XAY.

Theorem 2f. *Proof of* 2 *from* $A, B, C^3, 1^2, 8$.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By B, $XAY \smile AYX \smile YXA$. Suppose YXA. Then by 1, $BAX \,.\, AXY \,.\supset.\, BAY$, contrary to AYB, by C. Suppose AYX. Then by 8, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXA$. But BXA is contrary to XAB, by C; while if XBY, then by 1, $XBY \,.\, BYA \,.\supset.\, XBA$, contrary to XAB, by C. Therefore XAY.

Theorem 2g. *Proof of* 2 *from* $A, B^2, C^3, 1^3, 5$.

To prove: $XAB \,.\, AYB \,.\supset.\, XAY$. By C, BAY and XBA and BXA

are false, since AYB and XAB are true. By B, $AXY \smile XYA \smile XAY$; and by B, $XBY \smile BXY \smile BYX$.

Case 1. Suppose AXY. Then by 1, $BAX \,.\, AXY \,.\!\supset\!.\, BAY$, which is false.

Case 2. Suppose XBY. Then by 1, $XBY \,.\, BYA \,.\!\supset\!.\, XBA$, which is false.

Case 3. Suppose XYA and BXY. Then by 1, $BXY \,.\, XYA \,.\!\supset\!.\, BXA$, which is false.

Case 4. Suppose BYX. Then by 5, $BAX \,.\, BYX \,.\!\supset\!.\, BAY \smile YAX$, where BAY is false. Therefore XAY.

THEOREM 2h. *Proof of* 2 *from* A, C, 3, 8.

To prove: $XAB \,.\, AYB \,.\!\supset\!.\, XAY$. By 3, $XAB \,.\, AYB \,.\!\supset\!.\, XYB$. Hence by 8, $XYB \,.\, AYB \,.\!\supset\!.\, XAY \smile AXB$. But AXB is contrary to XAB, by C. Therefore XAY.

THEOREM 2i. *Proof of* 2 *from* A, C, 3, 5.

To prove: $XAB \,.\, AYB \,.\!\supset\!.\, XAY$. By 3, $XAB \,.\, AYB \,.\!\supset\!.\, XYB$. Hence by 5, $XAB \,.\, XYB \,.\!\supset\!.\, XAY \smile YAB$. But YAB is contrary to AYB, by C. Therefore XAY.

PROOFS OF POSTULATE 3

THEOREM 3a. *Proof of* 3 *from* A, B^2, C^3, 1^4.

To prove: $XAB \,.\, AYB \,.\!\supset\!.\, XYB$. By C, XBA and BAY and BXA are false, since XAB and AYB are true. By B, $XBY \smile BXY \smile XYB$; and by B, $AXY \smile XYA \smile YAX$.

Case 1. Suppose XBY. Then by 1, $XBY \,.\, BYA \,.\!\supset\!.\, XBA$, which is false.

Case 2. Suppose AXY. Then by 1, $BAX \,.\, AXY \,.\!\supset\!.\, BAY$, which is false.

Case 3. Suppose BXY and XYA. Then by 1, $BXY \,.\, XYA \,.\!\supset\!.\, BXA$, which is false.

Case 4. Suppose YAX. Then by 1, $BYA \,.\, YAX \,.\!\supset\!.\, BYX$. Therefore XYB.

THEOREM 3b. *Proof of* 3 *from* A, B, C, 2^3.

To prove: $XAB \,.\, AYB \,.\!\supset\!.\, XYB$. By B, $YBX \smile YXB \smile XYB$.

Case 1. Suppose YBX. Then by 2, $YBX \,.\, BAX \,.\!\supset\!.\, YBA$, contrary to AYB, by C.

Case 2. If YXB, then by 2, $YXB \,.\, XAB \,.\!\supset\!.\, YXA$, whence, by 2, $BYA \,.\, YXA \,.\!\supset\!.\, BYX$. Therefore XYB.

THEOREM 3c. *Proof of* 3 *from* A, C, 2^2, 6.

To prove: $XAB \,.\, AYB \,.\!\supset\!.\, XYB$. By 2, $XAB \,.\, AYB \,.\!\supset\!.\, XAY$.

Hence by 6, $BAX \,.\, YAX \,.\supset.\, BYX \smile YBX$. But if YBX, then by 2, $YBX \,.\, BAX \,.\supset.\, YBA$, contrary to AYB, by C. Therefore XYB.

THEOREM 3d. *Proof of* 3 *from A*, 1, 2.

To prove: $XAB \,.\, AYB \,.\supset.\, XYB$. By 2, $XAB \,.\, AYB \,.\supset.\, XAY$. Hence by 1, $BYA \,.\, YAX \,.\supset.\, BYX$. Therefore XYB.

THEOREM 3e. *Proof of* 3 *from A, C*, 2, 8.

To prove: $XAB \,.\, AYB \,.\supset.\, XYB$. By 2, $XAB \,.\, AYB \,.\supset.\, XAY$. Hence by 8, $YAX \,.\, BAX \,.\supset.\, YBA \smile BYX$. But YBA is contrary to AYB, by C. Therefore XYB.

PROOFS OF POSTULATE 4

THEOREM 4a. *Proof of* 4 *from A, B, C*, 1.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$. By B, $AXY \smile AYX \smile XAY$. Suppose XAY. Then by 1, $XAY \,.\, AYB \,.\supset.\, XAB$, contrary to AXB, by C. Therefore $AXY \smile AYX$.

THEOREM 4b. *Proof of* 4 *from A, B*, 1, 2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$. By B, $AXY \smile AYX \smile XAY$.
But if XAY, then by 1, $BYA \,.\, YAX \,.\supset.\, BYX$,
whence, by 2, $AXB \,.\, XYB \,.\supset.\, AXY$.
Therefore $AXY \smile AYX$.

THEOREM 4c. *Proof of* 4 *from A, B*, 1^2, 7.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$. By B, $AXY \smile AYX \smile YAX$.
But if XAY, then by 1, $XAY \,.\, AYB \,.\supset.\, XAB$;
and by 1, $YAX \,.\, AXB \,.\supset.\, YAB$;
whence by 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$. Therefore $AXY \smile AYX$.

THEOREM 4d. *Proof of* 4 *from A, C*, 5^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$;
and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.
Suppose AXY and AYX are both false. Then YXB and XYB, contrary to C.
Therefore $AXY \smile AYX$.

THEOREM 4e. *Proof of* 4 *from A*, 3^2, 5^2, 7^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$,
and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.
Suppose AXY and AYX are both false.
Then YXB and XYB, whence by 7, $XYB \,.\, AYB.\supset.\, AXY \smile XAY$.
But if XAY, then by 3, $BXY \,.\, XAY \,.\supset.\, BAY$,
and by 3, $BYX \,.\, YAX \,.\supset.\, BAX$;
whence by 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$. Therefore $AXY \smile AYX$.

THEOREM 4f. *Proof of* 4 *from* A, 5^2, 7, 8^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$,
and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.
Suppose AXY and AYX are both false.
Then YXB and XYB, whence by 8, $AXB \,.\, YXB.\supset.\, AYX \smile YAB$,
and by 8, $AYB \,.\, XYB \,.\supset.\, AXY \smile XAB$.
But if YAB and XAB, then by 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
Therefore $AXY \smile AYX$.

THEOREM 4g. *Proof of* 4 *from* 2, 5.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
But if YXB, then by 2, $AYB \,.\, YXB \,.\supset.\, AYX$. Therefore $AXY \smile AYX$.

THEOREM 4h. *Proof of* 4 *from* A, 1^2, 5, 7^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
Suppose AXY false.
Then YXB, whence, by 7, $YXB \,.\, AXB \,.\supset.\, YAX \smile AYX$.
But if YAX, then by 1, $YAX \,.\, AXB \,.\supset.\, YAB$,
and by 1, $XAY \,.\, AYB \,.\supset.\, XAB$,
whence by 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$. Therefore $AXY \smile AYX$.

The following two theorems, 4i and 4j, are the only ones in which Postulate C is used without Postulate A. (It will be noted that there are no cases in which Postulate B is used without Postulate A.)

THEOREM 4i. *Proof of* 4 *from* C^2, 5^2, 7^3, 8^2.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$,
and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.
Suppose AXY and AYX are both false. Then YXB and XYB,
whence by 7, $YXB \,.\, AXB \,.\supset.\, YAX \smile AYX$,
and by 7, $XYB \,.\, AYB \,.\supset.\, XAY \smile AXY$; whence $YAX$and XAY.
Again, by 8, $AXB \,.\, YXB \,.\supset.\, AYX \smile YAB$,
and by 8, $AYB \,.\, XYB \,.\supset.\, AXY \smile XAB$, whence YAB and XAB;
whence, by 7, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYA$, contrary to YAX and XAY, respectively, by C. Therefore $AXY \smile AYX$.

THEOREM 4j. *Proof of* 4 *from* C^2, 1^2, 5^2, 7^3.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
By 5, $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$,
and by 5, $AYB \,.\, AXB \,.\supset.\, AYX \smile XYB$.
Suppose AXY and AYX are both false. Then YXB and XYB,

whence by 7, $AXB \,.\, YXB \,.\supset.\, AYX \smile YAX$,
and by 7, $XYB \,.\, AYB \,.\supset.\, XAY \smile AXY$.
But if YAX and XAY, then by 1, $YAX \,.\, AXB \,.\supset.\, YAB$,
and by 1, $XAY \,.\, AYB \,.\supset.\, XAB$,
whence by 7, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYA$, contrary to YAX and XAY, respectively, by C. Therefore $AXY \smile AYX$.

Proofs of postulate 5

Theorem 5a. *Proof of 5 from A, B, 1, 2.*

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XYA \smile XAY$. But if XYA, then by 2, $BXA \,.\, XYA \,.\supset.\, BXY$; and if XAY, then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$. Therefore $AXY \smile YXB$.

Theorem 5b. *Proof of 5 from A, B, 1^2, 7.*

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XAY \smile XYA$.

Case 1. If XAY, then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$.

Case 2. If XYA, then by 7, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXY$. But if XBY, then by 1, $AXB \,.\, XBY \,.\supset.\, AXY$. Therefore $AXY \smile YXB$.

Theorem 5c. *Proof of 5 from A, B, C, 1, 8.*

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XAY \smile XYA$.

Case 1. Suppose XAY; then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$.

Case 2. Suppose XYA; then by 8, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBA$, where XBA is contrary to AXB, by C. Therefore $AXY \smile YXB$.

Theorem 5d. *Proof of 5 by A, B^2, C^3, 1^2, 6.*

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$. By B, $AXY \smile XAY \smile XYA$, and by B, $YXB \smile BYX \smile YBX$.

Case 1. If XAY, then by 1, $BXA \,.\, XAY \,.\supset.\, BXY$.

Case 2. If XYA and BYX, then by 6, $AYX \,.\, BYX \,.\supset.\, ABX \smile BAX$, contrary to AXB, by C.

Case 3. If XYA and YBX, then by 1, $YBX \,.\, BXA \,.\supset.\, YBA$, contrary to AYB, by C. Therefore $AXY \smile YXB$.

Theorem 5e. *Proof of 5 from A, 2, 4.*

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
But if AYX, then by 2, $BXA \,.\, XYA \,.\supset.\, BXY$. Therefore $AXY \smile YXB$.

Theorem 5f. *Proof of 5 from A, C, 4^2, 7.*

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$;
and by 4, $BXA \,.\, BYA \,.\supset.\, BXY \smile BYX$.
Suppose AYX and BYX. Then by 7, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXY$, where XBY is contrary to BYX, by C. Therefore $AXY \smile YXB$.

THEOREM 5g. *Proof of* 5 *from* A, C^2, 4^2, 6.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$;
and by 4, $BXA \,.\, BYA \,.\supset.\, BXY \smile BYX$.
Suppose AXY and YXB are both false.
Then AYX and BYX, whence by 6, $AYX \,.\, BYX \,.\supset.\, ABX \smile BAX$, contrary to AXB, by C. Therefore $AXY \smile YXB$.

THEOREM 5h. *Proof of* 5 *from* A, 1, 4, 7.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
If AYX, then by 7, $XYA \,.\, BYA \,.\supset.\, XBY \smile BXY$. But if XBY, then by 1, $AXB \,.\, XBY \,.\supset.\, AXY$. Therefore $AXY \smile YXB$.

THEOREM 5i. *Proof of* 5 *from* A, C, 4, 8.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
But if AYX, then by 8, $BYA \,.\, XYA \,.\supset.\, BXY \smile XBA$, where XBA is contrary to AXB, by C. Therefore $AXY \smile YXB$.

THEOREM 5j. *Proof of* 5 *from* A, 3^2, 4^2, 7.

To prove: $AXB \,.\, AYB \,.\supset.\, AXY \smile YXB$.
By 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$,
and by 4, $BXA \,.\, BYA \,.\supset.\, BXY \smile BYX$.
Suppose AXY and YXB are both false.
Then AYX and BYX, whence by 7, $AYX \,.\, BYX \,.\supset.\, ABY \smile BAY$.
But if ABY, then by 3, $YBA \,.\, BXA \,.\supset.\, YXA$; and if BAY, then by 3, $YAB \,.\, AXB \,.\supset.\, YXB$. Therefore $AXY \smile YXB$.

PROOFS OF POSTULATE 6

THEOREM 6a. *Proof of* 6 *from* A, B, C, 2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$. By B, $XYB \smile YBX \smile BXY$.
Suppose XBY. Then by 2, $XBY \,.\, BAY \,.\supset.\, XBA$, contrary to XAB, by C. Therefore $XYB \smile YXB$.

THEOREM 6b. *Proof of* 6 *from* A, B, 2^2, 7.

To prove: $XAB.\, YAB \,.\supset.\, XYB \smile YXB$. By B, $XYB \smile YBX \smile BXY$.
But if XBY, then by 2, $XBY \,.\, BAY \,.\supset.\, XBA$;
and by 2, $YBX \,.\, BAX \,.\supset.\, YBA$;
whence by 7, $XBA \,.\, YBA \,.\supset.\, XYB \smile YXB$. Therefore $XYB \smile YXB$.

THEOREM 6c. *Proof of* 6 *from* 1^2, 7.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
By 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

Case 1. If XYA, then by 1, $XYA \,.\, YAB \,.\supset.\, XYB$.

Case 2. If YXA, then by 1, $YXA \,.\, XAB \,.\supset.\, YXB$.
Therefore $XYB \smile YXB$.

THEOREM 6d. *Proof of* 6 *from* A, 3^2, 7.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
By 7, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.

Case 1. If XYA, then by 3, $BAX \,.\, AYX \,.\supset.\, BYX$.

Case 2. If YXA, then by 3, $BAY \,.\, AXY \,.\supset.\, BXY$.
Therefore $XYB \smile YXB$.

THEOREM 6e. *Proof of* 6 *from* 1, 8.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
By 8, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
But if XYA, then by 1, $XYA \,.\, YAB \,.\supset.\, XYB$.
Therefore $XYB \smile YXB$.

THEOREM 6f. *Proof of* 6 *from* A, 3, 8.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
By 8, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
But if XYA, then by 3, $BAX \,.\, AYX \,.\supset.\, BYX$.
Therefore $XYB \smile YXB$.

THEOREM 6g. *Proof of* 6 *from* A, B, 2^2, 8.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$. By B, $XYB \smile YXB \smile YBX$, and by 8, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$. But if YBX and XYA, then by 2, $YBX \,.\, BAX \,.\supset.\, YBA$, whence by 2, $XYA \,.\, YBA \,.\supset.\, XYB$. Therefore $XYB \smile YXB$.

THEOREM 6h. *Proof of* 6 *from* A, B, C^2, 3, 5.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$. By B, $XYB \smile YBX \smile BXY$. Suppose YBX. Then by 3, $YBX \,.\, BAX \,.\supset.\, YAX$, whence by 5,
$YBX \,.\, YAX \,.\supset.\, YBA \smile ABX$, contrary to YAB and XAB, respectively, by C. Therefore $XYB \smile YXB$.

THEOREM 6i. *Proof of* 6 *from* A, C, 8^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
By 8, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$;
and by 8, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYB$.
Suppose YXB and XYB are both false. Then XYA and YXA, contrary to C. Therefore $XYB \smile YXB$.

THEOREM 6j. *Proof of* 6 *from* A, B^2, C^2, 1^2, 5.

To prove: $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$. By C, XBA and ABY are false, since XAB and YAB are true. By B, $XAY \smile XYA \smile YXA$; and by B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XAY and XBY.
Then by 5, $XBY \,.\, XAY \,.\supset.\, XBA \smile ABY$, which are both false.

Case 2. If XYA, then by 1, $XYA \,.\, YAB \,.\!\supset\!.\, XYB$.

Case 3. If YXA, then by 1, $YXA \,.\, XAB \,.\!\supset\!.\, YXB$.

Therefore $XYB \smile YXB$.

Proofs of postulate 7

Theorem 7a. *Proof of* 7 *from A, B, C*, 2^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$. By B, $XYB \smile YXB \smile XBY$.

Case 1. If XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.

Case 2. If YXB, then by 2, $YXB \,.\, XAB \,.\!\supset\!.\, YXA$.

Case 3. If XBY, then by 2, $XBY \,.\, BAY \,.\!\supset\!.\, XBA$, contrary to XAB, by C. Therefore $XYA \smile YXA$.

Theorem 7b. *Proof of* 7 *from A, B*, C^3, 6^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$. Hence by C, YBX is false.

By B, $XAY \smile XYA \smile YXA$. Suppose XAY.

Then by 6, $YAX \,.\, BAX \,.\!\supset\!.\, YBX \smile BYX$;

and by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$. But YBX is false; hence both BYX and BXY must be true, which is impossible by C.

Therefore $XYA \smile YXA$.

Theorem 7c. *Proof of* 7 *from A*, C^3, 4^2, 6^3.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.

Case 1. Suppose XYB true. Then by 4, $XAB \,.\, XYB \,.\!\supset\!.\, XAY \smile XYA$. But if XAY, then by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$, contrary to XYB by C. Hence in Case 1, XYA.

Case 2. Suppose XYB false; then YXB.

Then by 4, $YAB \,.\, YXB \,.\!\supset\!.\, YAX \smile YXA$.

But if YAX, then by 6, $YAX \,.\, BAX \,.\!\supset\!.\, YBX \smile BYX$, where BYX is false by hypothesis, and YBX is contrary to YXB by C. Hence in Case 2, YXA.

Therefore $XYA \smile YXA$.

Theorem 7d. *Proof of* 7 *from* 2^2, 6.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.

Case 1. If XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.

Case 2. If YXB, then by 2, $YXB \,.\, XAB \,.\!\supset\!.\, YXA$.

Hence in either case, $XYA \smile YXA$.

Theorem 7e. *Proof of* 7 *from* 2, 8.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.

By 8, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.

If YXB, then by 2, $YXB \,.\, XAB \,.\!\supset\!.\, YXA$.

Therefore $XYA \smile YXA$.

THEOREM 7f. *Proof of* 7 *from* A, C, 8^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
By 8, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$;
and by 8, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYB$.
Suppose XYA and YXA are both false. Then YXB and XYB, contrary to C. Therefore $XYA \smile YXA$.

THEOREM 7g. *Proof of* 7 *from* A, B^2, C^4, 5^3.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$. By B, $XYB \smile YXB \smile XBY$, and by B, $XAY \smile XYA \smile YXA$.

Case 1. If XYB, then by 5, $XYB \,.\, XAB \,.\supset.\, XYA \smile AYB$, where AYB is contrary to YAB, by C. Hence in Case 1, XYA.

Case 2. If YXB, then by 5, $YXB \,.\, YAB \,.\supset.\, YXA \smile AXB$, where AXB is contrary to XAB, by C. Hence in Case 2, YXA.

Case 3. If XBY and XAY, then by 5, $XBY \,.\, XAY \,.\supset.\, XBA \smile ABY$, contrary to XAB and YAB, by C. Therefore $XYA \smile YXA$.

THEOREM 7h. *Proof of* 7 *from* A, C^2, 5^2, 6.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.

Case 1. If XYB, then by 5, $XYB \,.\, XAB \,.\supset.\, XYA \smile AYB$, where AYB is contrary to YAB, by C. Hence in Case 1, XYA.

Case 2. If YXB, then by 5, $YXB \,.\, YAB \,.\supset.\, YXA \smile AXB$, where AXB is contrary to XAB, by C. Hence in Case 2, YXA. Therefore $XYA \smile YXA$.

THEOREM 7i. *Proof of* 7 *from* A, 4, 5^2, 8^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
By 8, $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$;
and by 8, $YAB \,.\, XAB \,.\supset.\, YXA \smile XYB$.
Suppose XYA and YXA are both false; then YXB and XYB, whence by 5, $YXB \,.\, YAB \,.\supset.\, YXA \smile AXB$, and by 5, $XYB \,.\, XAB \,.\supset.\, XYA \smile AYB$.
But if AXB and AYB, then by 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$.
Therefore $XYA \smile YXA$.

THEOREM 7j. *Proof of* 7 *from* A, 1^2, 4^3, 5^2, 6^3.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXA$.
By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$.
If XYB is true, then by 4, $XYB \,.\, XAB \,.\supset.\, XYA \smile XAY$;
and by 5, $XYB \,.\, XAB \,.\supset.\, XYA \smile AYB$.
If YXB is true, then by 4, $YXB \,.\, YAB \,.\supset.\, YXA \smile YAX$,
and by 5, $YXB \,.\, YAB \,.\supset.\, YXA \smile AXB$. Suppose that XYA and YXA are both false. Then there are three cases to consider.

Case 1. Suppose XYB true and YXB false. Then XAY and AYB.

Then by 6, $XAY \,.\, BAY \,.\supset.\, XBY \smile BXY$,
whence by 1, $AYB \,.\, YBX \,.\supset.\, AYX$.

Case 2. Suppose XYB false and YXB true. Then YAX and AXB. Then by 6, $YAX \,.\, BAX \,.\supset.\, YBX \smile BYX$,
whence by 1, $AXB \,.\, XBY \,.\supset.\, AXY$.

Case 3. Suppose XYB true and YXB true. Then AYB and AXB. Then by 4, $AXB \,.\, AYB \,.\supset.\, AXY \smile AYX$. Therefore $XYA \smile YXA$.

Proofs of postulate 8

Theorem 8a. *Proof of* 8 *from A, B, C,* 2^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$. By B, $XBY \smile XYB \smile YXB$.

Case 1. If XBY, then by 2, $XBY \,.\, BAY \,.\supset.\, XBA$, contrary to XAB, by C.

Case 2. If XYB, then by 2, $XYB \,.\, YAB \,.\supset.\, XYA$.
Therefore $XYA \smile YXB$.

Theorem 8b. *Proof of* 8 *from A,* B^2, C^3, 1, 5^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$. By C, XBA and ABY and AYB are false, since XAB and YAB are true. By B, $XAY \smile YXA \smile XYA$; and by B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XAY and XBY.
Then by 5, $XBY \,.\, XAY \,.\supset.\, XBA \smile ABY$, which are both false.

Case 2. If XYB, then by 5, $XYB \,.\, XAB \,.\supset.\, XYA \smile AYB$, where AYB is false.

Case 3. If YXA, then by 1, $YXA \,.\, XAB \,.\supset.\, YXB$.
Therefore $XYA \smile YXB$.

Theorem 8c. *Proof of* 8 *from A,* B^2, C^3, 3, 5^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$. By C, XBA and ABY and AYB are false, since XAB and YAB are true. By B, $XAY \smile YXA \smile XYA$; and by B, $XBY \smile XYB \smile YXB$.

Case 1. Suppose XAY and XBY.
Then by 5, $XBY \,.\, XAY \,.\supset.\, XBA \smile ABY$, which are both false.

Case 2. If XYB, then by 5, $XYB \,.\, XAB \,.\supset.\, XYA \smile AYB$, where AYB is false.

Case 3. If YXA, then by 3, $BAY \,.\, AXY \,.\supset.\, BXY$.
Therefore $XYA \smile YXB$.

Theorem 8d. *Proof of* 8 *from A, B, C,* 3, 6^2.

To prove: $XAB \,.\, YAB \,.\supset.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\supset.\, XYB \smile YXB$; and by B, $AXY \smile XAY \smile XYA$.

Case 1. If AXY, then by 3, $BAY \,.\, AXY \,.\supset.\, BXY$. Hence in Case 1, YXB.

Case 2. If XAY and XYB, then by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$, where XBY is contrary to XYB, by C. Hence in Case 2, YXB. Therefore $XYA \smile YXB$.

THEOREM 8e. *Proof of* 8 *from* 1, 7.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.
By 7, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.
But if YXA, then by 1, $YXA \,.\, XAB \,.\!\supset\!.\, YXB$.
Therefore $XYA \smile YXB$.

THEOREM 8f. *Proof of* 8 *from* A, B, C, 1, 6^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$; and by B, $YXA \smile XAY \smile XYA$.

Case 1. If YXA, then by 1, $YXA \,.\, XAB \,.\!\supset\!.\, YXB$. Hence in Case 1, YXB.

Case 2. If XAY and XYB, then by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$, where XBY is contrary to XYB, by C. Hence in Case 2, YXB. Therefore $XYA \smile YXB$.

THEOREM 8g. *Proof of* 8 *from* A, 3, 7.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.
By 7, $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXA$.
But if YXA, then by 3, $BAY \,.\, AXY \,.\!\supset\!.\, BXY$.
Therefore $XYA \smile YXB$.

THEOREM 8h. *Proof of* 8 *from* 2, 6.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.
But if XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.
Therefore $XYA \smile YXB$.

THEOREM 8i. *Proof of* 8 *from* A, C, 5, 6.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.
But if XYB, then by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$, where AYB is contrary to YAB, by C. Therefore $XYA \smile YXB$.

THEOREM 8j. *Proof of* 8 *from* A, C, 4, 6^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$.
By 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$. Suppose XYB.
Then by 4, $XAB \,.\, XYB \,.\!\supset\!.\, XAY \smile XYA$. But if XAY, then by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$, where XBY is contrary to XYB, by C.
Therefore $XYA \smile YXB$.

THEOREM 8k. *Proof of* 8 *from* A, B, 2^3, 7.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. By B, $XBY \smile XYB \smile YXB$.

Case 1. If XBY, then by 2, $YBX \,.\, BAX \,.\!\supset\!.\, YBA$,

and by 2, $XBY \,.\, BAY \,.\!\supset\!.\, XBA$,
whence by 7, $XBA \,.\, YBA \,.\!\supset\!.\, XYB \smile YXB$.

Case 2. If XYB, then by 2, $XYB \,.\, YAB \,.\!\supset\!.\, XYA$.
Therefore $XYA \smile YXB$.

THEOREM 8l. *Proof of* 8 *from* A, B, 1^2, 5, 6^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. By B, $YXA \smile XAY \smile XYA$, and by 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$.

Case 1. If YXA, then by 1, $YXA \,.\, XAB \,.\!\supset\!.\, YXB$.

Case 2. If XAY and XYB, then by 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$, and by 6, $XAY \,.\, BAY \,.\!\supset\!.\, XBY \smile BXY$. But if AYB and XBY, then by 1, $AYB \,.\, YBX \,.\!\supset\!.\, AYX$. Therefore $XYA \smile YXB$.

THEOREM 8m. *Proof of* 8 *from* A, 1, 4, 5, 6^2.

To prove: $XAB \,.\, YAB \,.\!\supset\!.\, XYA \smile YXB$. Suppose XYA and YXB are both false. Now by 6, $XAB \,.\, YAB \,.\!\supset\!.\, XYB \smile YXB$. As YXB is false, XYB. By 4, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile XAY$. As XYA is false, XAY. By 5, $XYB \,.\, XAB \,.\!\supset\!.\, XYA \smile AYB$. As XYA is false, AYB. By 6, $BAY \,.\, XAY \,.\!\supset\!.\, BXY \smile XBY$. As BXY is false, then XBY. By 1, $AYB \,.\, YBX \,.\!\supset\!.\, AYX$, contrary to supposition. Therefore $XYA \smile YXB$.

The results of these 71 theorems may be conveniently summarized in the following table. In this table, the numbers in the last column indicate the sets of independent postulates, if any (see § 4), in connection with which each theorem is available.

TABLE I. THEOREMS ON DEDUCIBILITY

Theorem	Postulate	follows from											Set.
1a	1	A	B	C		2		4					6
1b	1	A	B	C			3	4					9, 10, 11
1c	1	A	B	C		2			5				7
1d	1	A	B	C			3		5				8
2a	2	A	B	C	1						7		4
2b	2	A	B	C	1					6			3
2c	2	A	B	C			3			6			9
2d	2	A		C			3				7		10
2e	2	A		C			3	4		6			9
2f	2	A	B	C	1							8	5
2g	2	A	B	C	1				5				2
2h	2	A		C			3					8	11
2i	2	A		C			3		5				8
3a	3	A	B	C	1								1, 2, 3, 4, 5
3b	3	A	B	C		2							1, 6, 7
3c	3	A		C		2				6			
3d	3	A			1	2							1
3e	3	A		C		2						8	
4a	4	A	B	C	1								1, 2, 3, 4, 5
4b	4	A	B		1	2							1
4c	4	A	B		1						7		4
4d	4	A		C					5				2, 7, 8
4e	4	A					3		5		7		

TABLE I—*Continued*

Theorem	Postulate	follows from											Set.
4f	4	A							5		7	8	
4g	4					2			5				7
4h	4	A			1				5		7		
4i	4			C					5		7	8	
4j	4			C	1				5		7		
5a	5	A	B		1	2							1
5b	5	A	B		1						7		4
5c	5	A	B	C	1							8	5
5d	5	A	B	C	1					6			3
5e	5	A				2		4					6
5f	5	A		C				4			7		10
5g	5	A		C				4		6			9
5h	5	A			1			4			7		
5i	5	A		C				4				8	11
5j	5	A					3	4			7		10
6a	6	A	B	C		2							1, 6, 7
6b	6	A	B			2					7		
6c	6				1						7		4
6d	6	A					3				7		10
6e	6				1							8	5
6f	6	A					3					8	11
6g	6	A	B			2						8	
6h	6	A	B	C			3		5				8
6i	6	A		C								8	5, 11
6j	6	A	B	C	1				5				2
7a	7	A	B	C		2							1, 6, 7
7b	7	A	B	C						6			3, 9
7c	7	A		C				4		6			9
7d	7					2				6			
7e	7					2						8	
7f	7	A		C								8	5, 11
7g	7	A	B	C					5				2, 7, 8
7h	7	A		C					5	6			
7i	7	A						4	5			8	
7j	7	A			1			4	5	6			
8a	8	A	B	C		2							1, 6, 7
8b	8	A	B	C	1				5				2
8c	8	A	B	C			3		5				8
8d	8	A	B	C			3			6			9
8e	8				1						7		4
8f	8	A	B	C	1					6			3
8g	8	A					3				7		10
8h	8					2				6			
8i	8	A		C					5	6			
8j	8	A		C				4		6			9
8k	8	A	B			2					7		
8l	8	A	B		1				5	6			
8m	8	A			1			4	5	6			

3. Theorems on non-deducibility and examples of pseudo-betweenness

In this section we show that the question proposed at the beginning of § 2 must always be answered in the negative, except in the cases covered by the 71 theorems just established. That is, we show that no one of the twelve postulates of our basic list is deducible from any others of the list, except in the cases covered by our 71 theorems.

In order to prove this statement, we first construct 44 *examples of pseudo-betweenness*, that is, 44 examples of systems (K, R) which satisfy some but not all of the twelve postulates.

In the first four examples, A–D, the class K consists of three elements.

Example A. Let K = a class of three numbers, say 1, 2, 3, and let XYZ be true in the cases 123, 231, and false in all other cases.

Here 123 is true, while 321 is false, so that Postulate A is not satisfied. C is satisfied vacuously, since the conditions mentioned in the hypothesis do not occur. B and D hold. Postulates 1–8 are satisfied vacuously, since the class contains only three elements.

Example B. Let K = a class of three numbers, say 1, 2, 3, and let XYZ be false for all values of X, Y, Z.

Here B is clearly not satisfied. A, C, D, and 1–8 are satisfied vacuously.

Example C. Let K = a class of three numbers, say 1, 2, 3, and let XYZ mean that X, Y, Z are distinct.

Here A, B, and D are satisfied, while C is not. Postulates 1–8 are satisfied vacuously.

Example D. Let K = a class of any three numbers; and let XYZ mean that Y belongs to the interval from X to Z inclusive, when X, Y, and Z are arranged in order of magnitude.

Here A, B, and C are satisfied, while D is not. Postulates 1–8 are satisfied vacuously.

In the remaining examples, 1–40, the class K consists of four numbers, 1, 2, 3, 4, and the meaning of $\mathrm{R}\,[XYZ]$ is defined by simply giving a catalog of the ordered triads of elements for which the relation is true.

In all these examples, Postulate D is satisfied.* Which of the other postulates is satisfied in each case may be ascertained from Table II.

In examples 1–5, all four of the Postulates A, B, C, D are satisfied.

Example 1. 124, 134, 213, 243, 312, 342, 421, 431.

Example 2. 123, 142, 234, 241, 314, 321, 413, 432.

Example 3. 123, 142, 143, 241, 243, 321, 341, 342.

Example 4. 142, 213, 234, 241, 312, 314, 413, 432.

Example 5. 123, 143, 214, 321, 324, 341, 412, 423.

In Examples 6–11, Postulate B fails, while A, C, and D hold.

Example 6. 123, 234, 321, 432.

Example 7. 123, 143, 243, 321, 341, 342.

Example 8. 123, 243, 321, 342.

Example 9. 123, 124, 243, 321, 342, 421.

* In verifying these examples with respect to Postulates 5 and 8, the following peculiarity of these two postulates should be borne in mind: as to 5, for example, it is not sufficient to test for $AXB \cdot AYB$; it is necessary also to test for $AYB \cdot AXB$; and similarly as to 8. For illustrations of the importance of this precaution, see, for instance, Examples 33 and 34.

EXAMPLE 10. 123, 143, 321, 341.

EXAMPLE 11. 213, 214, 312, 412.

In Examples 12–24, Postulate C fails, while A, B, and D hold.

EXAMPLE 12. 123, 132, 214, 231, 234, 314, 321, 324, 412, 413, 423, 432.

EXAMPLE 13. 124, 132, 134, 213, 214, 231, 234, 312, 314, 324, 412, 413, 421, 423, 431, 432.

EXAMPLE 14. 124, 213, 234, 312, 314, 324, 413, 421, 423, 432.

EXAMPLE 15. 123, 124, 143, 234, 243, 321, 341, 342, 421, 432.

EXAMPLE 16. 123, 143, 213, 214, 243, 312, 314, 321, 324, 341, 342, 412, 413, 423.

EXAMPLE 17. 123, 143, 214, 243, 321, 324, 341, 342, 412, 423.

EXAMPLE 18. 213, 214, 234, 243, 312, 314, 324, 342, 412, 413, 423, 432.

EXAMPLE 19. 123, 124, 132, 134, 231, 234, 321, 324, 421, 423, 431, 432.

EXAMPLE 20. 123, 124, 134, 143, 234, 243, 321, 324, 341, 342, 421, 423, 431, 432.

EXAMPLE 21. 123, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 321, 341, 342, 412, 431, 432.

EXAMPLE 22. 123, 124, 132, 134, 143, 214, 231, 243, 321, 324, 341, 342, 412, 421, 423, 431.

EXAMPLE 23. 123, 124, 134, 143, 321, 324, 341, 421, 423, 431.

EXAMPLE 24. 123, 134, 142, 214, 241, 321, 324, 412, 423, 431.

In Examples 25–37, Postulate A fails, while B, C, and D hold.

EXAMPLE 25. 123, 142, 234, 341.

EXAMPLE 26. 123, 142, 143, 213, 214, 243, 413, 423, 421.

EXAMPLE 27. 123, 124, 243, 341.

EXAMPLE 28. 123, 143, 324, 421.

EXAMPLE 29. 123, 143, 213, 243, 412, 413, 423.

EXAMPLE 30. 123, 124, 143, 213, 214, 243, 413, 421, 423.

EXAMPLE 31. 123, 214, 341, 423.

EXAMPLE 32. 123, 142, 314, 412, 423.

EXAMPLE 33. 123, 142, 143, 324.

EXAMPLE 34. 123, 314, 412, 423.

EXAMPLE 35. 123, 124, 143, 243, 412, 423.

EXAMPLE 36. 123, 124, 143, 243, 423.

EXAMPLE 37. 123, 143, 214, 243, 412, 423.

In Example 38, Postulates A and C fail, while B and D hold.

EXAMPLE 38. 123, 143, 213, 214, 243, 412, 413, 421, 423.

In Examples 39 and 40, Postulates B and C fail, while A and D hold.

EXAMPLE 39. 123, 142, 214, 241, 321, 324, 412, 423.

EXAMPLE 40. 123, 143, 243, 321, 324, 341, 342, 423.

The properties possessed by these 44 systems are conveniently exhibited in Table II, in which a dot (.) indicates that a postulate is satisfied, while a cross (×) indicates that it is not satisfied.

TABLE II. EXAMPLES OF PSEUDO-BETWEENNESS

Ex.	A	B	C	D	1	2	3	4	5	6	7	8	Lemma in which used.
A	×	.	.	.	.	.	.	.	.	.	.	.	A′
B	.	×	.	.	.	.	.	.	.	.	.	.	B′
C	.	.	×	.	.	.	.	.	.	.	.	.	C′
D	.	.	.	×	.	.	.	.	.	.	.	.	D′
1	.	.	.	.	×	.	.	×	×	.	.	.	1′a, 4′a, 5′b.
2	.	.	.	.	×	×	×	.	.	.	.	.	1′b, 2′b, 3′a.
3	.	.	.	.	.	×	.	.	×	×	×	×	2′a, 5′a, 6′a, 7′a, 8′a.
4	.	.	.	.	×	×	×	.	.	×	.	×	6′b, 8′c.
5	.	.	.	.	×	×	×	×	×	.	.	×	8′b.
6	.	×	.	.	×	.	.	.	.	.	.	.	1′c.
7	.	×	.	.	.	×	.	×	×	.	×	×	2′c, 7′b, 8′f.
8	.	×	.	.	.	×	×	.	.	.	.	.	2′d, 3′b.
9	.	×	.	.	×	.	×	.	.	×	.	×	3′c, 6′d, 8′e.
10	.	×	.	.	.	.	.	×	×	.	.	.	4′b, 5′c.
11	.	×	.	.	.	.	.	.	.	×	×	×	6′c, 7′c, 8′d.
12	.	.	×	.	×	.	.	.	.	.	.	.	1′d.
13	.	.	×	.	.	×	.	.	.	.	.	.	2′e.
14	.	.	×	.	.	×	×	.	.	.	.	.	3′d.
15	.	.	×	.	×	.	×	.	.	.	.	.	3′e.
16	.	.	×	.	.	×	.	×	.	.	×	.	4′c, 7′d.
17	.	.	×	.	×	×	×	×	.	.	.	×	4′d.
18	.	.	×	.	.	×	.	.	×	.	×	.	5′d, 7′e.
19	.	.	×	.	.	.	.	.	.	×	×	×	6′e, 7′f, 8′g.
20	.	.	×	.	×	×	.	.	.	.	×	×	7′g, 8′h.
21	.	.	×	.	×	×	×	.	×	.	.	.	5′e.
22	.	.	×	.	×	×	×	.	.	.	.	×	8′i.
23	.	.	×	.	.	×	.	.	×	.	×	×	8′j.
24	.	.	×	.	×	×	×	.	.	×	.	.	6′f.
25	×	.	.	.	×	.	.	.	.	.	.	.	1′e.
26	×	.	.	.	.	×	.	.	.	.	.	.	2′f.
27	×	.	.	.	.	.	×	.	.	.	.	.	3′f.
28	×	.	.	.	.	.	.	×	×	.	.	.	4′e.
29	×	.	.	.	.	×	.	×	.	.	×	.	4′f.
30	×	.	.	.	.	×	.	.	.	.	×	.	7′h.
31	×	.	.	.	.	.	.	.	.	×	×	×	6′h, 7′i, 8′o.
32	×	.	.	.	×	.	.	.	.	×	.	.	6′i.
33	×	.	.	.	.	.	.	.	×	.	.	.	5′f.
34	×	.	.	.	×	.	.	.	.	×	.	×	8′l.
35	×	.	.	.	×	×	.	.	.	.	.	×	8′m.
36	×	.	.	.	.	×	.	.	.	.	×	×	8′n.
37	×	.	.	.	×	×	.	×	.	.	.	×	4′g.
38	×	.	×	.	.	×	.	×	.	.	.	.	4′h.
39	.	×	×	.	×	.	×	.	.	×	.	.	6′g.
40	.	×	×	.	.	×	.	×	.	.	×	×	8′k.

By inspection of these 44 examples, we have at once 68 lemmas on non-deducibility, as exhibited in Table III.

TABLE III. LEMMAS ON NON-DEDUCIBILITY

Lemma	Postulate	is not deducible from postulates												Proof by example
A′	A		B	C	D	1	2	3	4	5	6	7	8	A
B′	B	A		C	D	1	2	3	4	5	6	7	8	B
C′	C	A	B		D	1	2	3	4	5	6	7	8	C
D′	D	A	B	C		1	2	3	4	5	6	7	8	D
1′a	1	A	B	C	D		2	3			6	7	8	1
1′b	1	A	B	C	D				4	5	6	7	8	2
1′c	1	A		C	D		2	3	4	5	6	7	8	6
1′d	1	A	B		D		2	3	4	5	6	7	8	12
1′e	1		B	C	D		2	3	4	5	6	7	8	25
2′a	2	A	B	C	D	1		3	4					3
2′b	2	A	B	C	D				4	5	6	7	8	2
2′c	2	A		C	D	1		3			6			7
2′d	2	A		C	D	1			4	5	6	7	8	8
2′e	2	A	B		D	1		3	4	5	6	7	8	13
2′f	2		B	C	D	1		3	4	5	6	7	8	26
3′a	3	A	B	C	D				4	5	6	7	8	2
3′b	3	A		C	D	1			4	5	6	7	8	8
3′c	3	A		C	D		2		4	5		7		9
3′d	3	A	B		D	1			4	5	6	7	8	14
3′e	3	A	B		D		2		4	5	6	7	8	15
3′f	3		B	C	D	1	2		4	5	6	7	8	27
4′a	4	A	B	C	D		2	3			6	7	8	1
4′b	4	A		C	D	1	2	3			6	7	8	10
4′c	4	A	B		D	1		3		5	6		8	16
4′d	4	A	B		D					5	6	7		17
4′e	4		B	C	D	1	2	3			6	7	8	28
4′f	4		B	C	D	1		3		5	6		8	29
4′g	4		B	C	D			3		5	6	7		37
4′h	4		B		D	1		3		5	6	7	8	38
5′a	5	A	B	C	D	1		3	4					3
5′b	5	A	B	C	D		2	3			6	7	8	1
5′c	5	A		C	D	1	2	3			6	7	8	10
5′d	5	A	B		D	1		3	4		6		8	18
5′e	5	A	B		D				4		6	7	8	21
5′f	5		B	C	D	1	2	3	4		6	7	8	33
6′a	6	A	B	C	D	1		3	4					3
6′b	6	A	B	C	D				4	5		7		4
6′c	6	A		C	D	1	2	3	4	5				11
6′d	6	A		C	D		2		4	5		7		9
6′e	6	A	B		D	1	2	3	4	5				19
6′f	6	A	B		D				4	5		7	8	24
6′g	6	A			D		2		4	5		7	8	39
6′h	6		B	C	D	1	2	3	4	5				31
6′i	6		B	C	D		2	3	4	5		7	8	32
7′a	7	A	B	C	D	1		3	4					3
7′b	7	A		C	D	1		3			6			7
7′c	7	A		C	D	1	2	3	4	5				11
7′d	7	A	B		D	1		3		5	6		8	16
7′e	7	A	B		D	1		3	4		6		8	18
7′f	7	A	B		D	1	2	3	4	5				19
7′g	7	A	B		D			3	4	5	6			20
7′h	7		B	C	D	1		3	4	5	6		8	30
7′i	7		B	C	D	1	2	3	4	5				31

TABLE III—*Continued.*

Lemma	Postulate	is not deducible from postulates	Proof by example
8′a	8	A B C D 1 3 4	3
8′b	8	A B C D 6 7	5
8′c	8	A B C D 4 5 7	4
8′d	8	A C D 1 2 3 4 5	11
8′e	8	A C D 2 4 5 7	9
8′f	8	A C D 1 3 6	7
8′g	8	A B D 1 2 3 4 5	19
8′h	8	A B D 3 4 5 6	20
8′i	8	A B D 4 5 6 7	22
8′j	8	A B D 1 3 4 6	23
8′k	8	A D 1 3 5 6	40
8′l	8	B C D 2 3 4 5 7	34
8′m	8	B C D 3 4 5 6 7	35
8′n	8	B C D 1 3 4 5 6	36
8′o	8	B C D 1 2 3 4 5	31

By comparing these lemmas with the theorems in § 2, we can now establish the following theorem of non-deducibility:

THEOREM. *No one of the twelve postulates of our basic list is deducible from any others of the list, except in the cases covered by our* 71 *theorems.*

For example, consider the case of Postulate 3. By Lemmas 3′a–3′f, we see that Postulate 3 can certainly not be proved without the use of at least one postulate from each of the following groups: 1, 2; B, 2; B, 1, 6, 8; C, 2; C, 1; A; hence the following combinations are the only ones which need to be investigated: A, 1, 2; A, B, C, 1; A, B, C, 2; A, C, 2, 6; A, C, 2, 8. But by reference to Theorems 3a–3e we see that each one of these combinations is in fact sufficient to prove Postulate 3.

The truth of the theorem for each of the other cases is established in a similar way.

4. Eleven sets of independent postulates

We are now in position to select from our basic list of twelve postulates, several smaller lists which are free from redundancies.

An examination of our results in regard to deducibility shows that this selection can be made in precisely eleven ways; that is, there are precisely eleven sets of independent postulates which can be selected from our basic list.

The eleven sets are as follows:

(1) A, B, C, D, 1, 2. (6) A, B, C, D, 2, 4.
(2) A, B, C, D, 1, 5. (7) A, B, C, D, 2, 5.
(3) A, B, C, D, 1, 6. (8) A, B, C, D, 3, 5.
(4) A, B, C, D, 1, 7. (9) A, B, C, D, 3, 4, 6.
(5) A, B, C, D, 1, 8. (10) A, B, C, D, 3, 4, 7.
(11) A, B, C, D, 3, 4, 8.

That the postulates of each set are independent of one another is proved by the existence of Examples A–D, 1, 2, 3 above. (See Lemmas A–D, 1′a, 2′a, 3′a, 4′a, 5′a, 6′a, 7′a, 8′a, 1′b, 2′b, and 5′b.)

That the postulates of each set are sufficient to establish the entire list of twelve postulates is proved by our theorems on deducibility; in fact, in several cases the missing postulates can be deduced from the given postulates in more than one way. Table I, at the end of § 2, will show clearly all the possible ways in which the missing postulates in each set can be deduced from the given postulates of that set.

It will be noticed that certain theorems are not directly available in any of the eleven sets, since no one of the sets contains explicitly the postulates used in the proof of these theorems.

A comparison of the merits of the eleven sets of postulates by the aid of Table I, while perhaps not convincing in the present state of our knowledge of the standards to which such sets of postulates should conform, would at any rate be of some interest.

For example, if our aim is to find the set which shall be the most condensed, and from which the remaining postulates can be most readily deduced, we should select Set 1. If, on the other hand, our aim is to analyze the postulates down to their lowest terms, that is, to find a set from which the necessary deductions can just barely be made, we should then probably select Set 10. It is quite possible, of course, that some other considerations (not now clear) might lead us to select some other of the eleven sets as preferable for some purpose then in view.

In any case, it is satisfactory to know that these eleven sets are the only sets of independent postulates which can be selected from the basic list of twelve postulates from which we started.

Moreover, this basic list of twelve postulates must always occupy a central place in any theory of betweenness. For, as we have already pointed out, this set contains all the general laws concerning the betweenness relations among three or four elements; and even if further propositions concerning five or more distinct elements should be added to the list, no one of the basic list of twelve could thereby be made redundant. To prove this fact, we have merely to notice that the system exhibited in each of the examples used above in proving independence contains at most four elements, and would therefore satisfy vacuously any proposition involving five or more distinct elements.

5. Definition of betweenness

The following definition of betweenness may now be formulated:

Definition. Any system (K, R) in which the class K and the triadic relation R are found to possess all the properties demanded by any one of

our sets of independent postulates (see § 4) may be called an *ordered class* or *series*, and the relation R itself may then be called the relation of *betweenness*.

The most familiar example of such an ordered class or series is the system (K, R) in which K is the class of points on a line, and AXB means that the point X belongs to the interior of the segment AB.

Another example is the system (K, R) in which K is the class of natural numbers and AXB means that the number X is larger than the smaller of the two numbers A and B, and smaller than the larger one.

In each of these examples we say that X is "between" A and B.

The relation between the theory of betweenness and the theory of serial order may be expressed as follows.

Let A and B be any two distinct elements of a "betweenness" system, and let X and Y be any other distinct elements of the system. Then we say that *X precedes Y, in the order AB*, if any one of the following conditions is true: (1) XAB and either XYA or $Y = A$ or AYB or $Y = B$ or ABY; (2) $X = A$ and either AYB or $Y = B$ or ABY; (3) AXB and either XYB or $Y = B$ or ABY; (4) $X = B$ and ABY; (5) ABX and BXY. From this definition, and the properties of betweenness, it is easy to derive the usual properties of the dyadic relation of serial order, always, however, with respect to the fixed base AB.

On the relation between the theory of betweenness and the theory of cyclic order, see a paper by E. V. Huntington: *A set of independent postulates for cyclic order,* Proceedings of the National Academy of Sciences, vol. 2 (1916), p. 630.

Appendix

Remark Concerning Postulate A.—In regard to the general laws of betweenness concerning four elements A, B, X, Y on a line, if we agree to read always in the direction from A towards B, the total number of these general laws appears at first sight to be twenty-four, which group themselves into nine groups, as follows:

$-X$——A———B——Y——	$-Y$——A———$B-X$———
1. $XAB \,.\, ABY \,.\supset .\, XAY$	1a. $YAB \,.\, ABX \,.\supset .\, YAX$
1c. $XAB \,.\, ABY \,.\supset .\, XBY$	1b. $YAB \,.\, ABX \,.\supset .\, YBX$
$-X$——$A-Y$——B————	$-Y$——$A-X$——B————
2. $XAB \,.\, AYB \,.\supset .\, XAY$	2a. $YAB \,.\, AXB \,.\supset .\, YAX$
3. $XAB \,.\, AYB \,.\supset .\, XYB$	3a. $YAB \,.\, AXB \,.\supset .\, YXB$
———$A-X$——$B-Y$———	———$A-Y$——$B-X$———
2c. $AXB \,.\, ABY \,.\supset .\, XBY$	2b. $AYB \,.\, ABX \,.\supset .\, YBX$
3c. $AXB \,.\, ABY \,.\supset .\, AXY$	3b. $AYB \,.\, ABX \,.\supset .\, AYX$

4. $AXB\ .\ AYB\ .\supset.\ AXY \smile AYX$
4b. $AXB\ .\ AYB\ .\supset.\ XYB \smile YXB$

$$\text{———}A-\begin{matrix}X & Y\\ Y & X\end{matrix}-B\text{———}$$

5. $AXB\ .\ AYB\ .\supset.\ AXY \smile YXB$ 5a. $AXB\ .\ AYB\ .\supset.\ AYX \smile XYB$

6. $XAB\ .\ YAB\ .\supset.\ XYB \smile YXB$
7. $XAB\ .\ YAB\ .\supset.\ XYA \smile YXA$

$$-\begin{matrix}X & Y\\ Y & X\end{matrix}-A\text{———}B\text{———}$$

8. $XAB\ .\ YAB\ .\supset.\ XYA \smile YXB$ 8a. $XAB\ .\ YAB\ .\supset.\ YXA \smile XYB$

6b. $ABX\ .\ ABY\ .\supset.\ AXY \smile AYX$
7b. $ABX\ .\ ABY\ .\supset.\ BXY \smile BYX$

$$\text{———}A\text{———}B-\begin{matrix}X & Y\\ Y & X\end{matrix}-$$

8b. $ABX\ .\ ABY\ .\supset.\ AXY \smile BYX$ 8c. $ABX\ .\ ABY\ .\supset.\ AYX \smile BXY$

But each of the postulates in the second column is immediately obtainable from the postulate standing opposite it in the first column, without the use of any other postulate, so that the list of 24 is at once reducible to 15.

Furthermore, any two of the 24 postulates which bear the same number are deducible from each other by the aid of Postulate A alone. Hence the list of 24 reduces to 8, which may be selected in various ways; all these selections are equivalent in view of Postulate A; the Postulates 1–8 of the text represent one such selection.

On the other hand, if we agree to read either forward or backward along the line, the list of 24 would have to be greatly enlarged, so as to include, for example, such postulates as $XAB\ .\ YBA\ .\supset.\ YAX$. All such postulates are immediately deducible from Postulates 1–8 by the aid of Postulate A, and are not here considered. It should be noted, however, that if it were desired to give a complete discussion of what could be proved without the aid of Postulate A, it would be necessary to consider the whole of the enlarged list, and also to modify slightly the wording of Postulate C.

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HASKINS'S MOMENTAL THEOREM AND ITS CONNECTION WITH STIELTJES'S PROBLEM OF MOMENTS*

BY

EDWARD B. VAN VLECK

In the last volume of the Transactions † a simple and elegant proof of Haskins's Momental Theorem was given by Dunham Jackson through use of polynomial approximation. This interesting theorem is as follows: If for two bounded functions $f(x)$ and $\phi(x)$ integrable in the sense of Lebesgue for $a \leqq x \leqq b$ we have

$$(1) \qquad \int_a^b [f(x)]^n\,dx = \int_a^b [\phi(x)]^n\,dx \qquad (n = 1, 2, \cdots),$$

then the measures of the sets of points $E_{\alpha\beta}(f)$ and $E_{\alpha\beta}(\phi)$ for which the values of $f(x)$ and $\phi(x)$ respectively lie between α and β inclusive will be equal for all pairs of numbers (α, β) included between the upper and lower bounds of the functions; i. e.,

$$(2) \qquad mE_{\alpha\beta}(f) = mE_{\alpha\beta}(\phi).$$

A proof (unpublished) of the theorem with very "considerable restrictions" on the functions had been previously obtained by Haskins‡ "by reduction to the theorem of Stieltjes and Lebesgue concerning the moments." That theorem asserts that if for two bounded functions we have

$$(3) \qquad \int_a^b x^n f(x)\,dx = \int_a^b x^n \phi(x)\,dx \qquad (n = 0, 1, 2, \cdots),$$

then $f(x)$ and $\phi(x)$ are equal "almost everywhere" in the interval of integration, or, in other words, are equal except for at most a set of points of measure 0.§

In the following pages it is shown that a general proof of Haskins's theorem

* Presented to the Society, December 22, 1916.

† Vol. 17 (1916), p. 178.

‡ Cf. Haskins, these Transactions, vol. 17 (1916), p. 185, footnote.

§ The identity of $f(x)$ and $\phi(x)$, when continuous, was pointed out by Stieltjes in correspondence with Hermite in 1893, and the above extension was given by Lebesgue in 1909. For literature references see Haskins, p. 185.

can be obtained by reduction to Stieltjes's Theorem of Moments when the latter is expressed more comprehensively in terms of Stieltjes integrals. Thus in place of (3) we have as our basal set of equations one of the form

$$\int_a^b x^n\, df(x) = \int_a^b x^n\, d\phi(x) \qquad (n = 0, 1, 2, \cdots), \tag{4}$$

where the two functions are of limited variations and vanish at a. A very simple proof, closely resembling Jackson's proof of (2), is then given that $f(x)$ and $\phi(x)$ coincide throughout (a, b) except possibly at their points of discontinuity. It is also shown how this result may be obtained, though not so directly, from work of Stieltjes.

Incidentally it appears that every Lebesgue integral can be thrown into the form of a Stieltjes integral. It seems altogether likely that this may be known to some, but I have not found it stated. Lebesgue* has noted that every Stieltjes integral $\int f(x)\, d\alpha(x)$ containing a continuous $f(x)$ and an $\alpha(x)$ of limited variation can be transformed into a Lebesgue integral and, in fact, into a Riemannian integral.

By hypothesis let $f(x)$ and $\phi(x)$ be two functions bounded in the interval (a, b) of integration so that we have

$$h \leqq f(x) \leqq H, \qquad h \leqq \phi(x) \leqq H. \tag{5}$$

Place

$$\psi_f(y) = mE_{hy}(f) \equiv mE(h \leqq f(x) \leqq y),$$

where y is any value in the function range (h, H). This function† may be called the measure function of $f(x)$. It is obviously a monotone, non-decreasing function, continuous on the right-hand side. Now by the definition of the Lebesgue integral we have

$$\int_a^b [f(x)]^n\, dx = \lim \sum_h^H y_i^n \cdot mE(y_{i-1} < f(x) \leqq y_i) + h^n\, mE(f(x) = h). \tag{6}$$

But $mE(y_{i-1} < f(x) \leqq y_i)$ is the increment of ψ_f when the argument increases from y_{i-1} to y_i. Consequently we have the identity

$$\int_a^b [f(x)]^n\, dx = \int_h^H y^n\, d\psi_f(y), \tag{7}$$

provided we suppose the value of $\psi_f(y)$ to be altered at h and made there equal to 0, if necessary, so as to include in this Stieltjes integral the last member on the right of (6).

* Comptes Rendus, vol. 150 (1910), p. 86.

† Compare with Haskins's function M_y, loc. cit., p. 184.

For $n = 1$ the last equation converts a Lebesgue integral into a Stieltjes integral. The conversion is also applicable when $f(x)$ is an unlimited integrable function if we put $H = +\infty$, $h = -\infty$.

By virtue of (7) our hypothesis (1) now takes the form

$$(8) \qquad \int_h^H y^n \, d\psi_f(y) = \int_h^H y^n \, d\psi_\phi(y) \qquad (n = 0, 1, 2, \cdots).$$

Consider any two limited, non-decreasing functions $\psi_1(x)$ and $\psi_2(x)$, for which we have

$$(9) \qquad \psi_1(h) = \psi_2(h) = 0, \quad \int_h^H x^n \, d\psi_1(x) = \int_h^H x^n \, d\psi_2(x) \qquad (n = 0, 1, 2, \cdots).$$

The points of discontinuity of either function are countable, and the value of the function at each such point (h and H excepted) may be changed to the right-hand limiting value without affecting the value of the integral. We suppose for the moment this change made for both functions. If $P_n(x)$ is any polynomial, we have by (9)

$$(10) \qquad \int_h^H P_n(x) \, d\psi_1(x) = \int_h^H P_n(x) \, d\psi_2(x).$$

Take any fixed point x' in the interval of integration and construct a function $F(x)$ equal to 1 for $h \leqq x \leqq x'$, equal between x' and $x' + \zeta$ to a linear function which decreases from 1 to 0, and equal to 0 for $x' + \zeta \leqq x \leqq H$. Since $F(x)$ is continuous, a polynomial exists such that throughout the interval (h, H) we have $|F(x) - P_n(x)| \leqq \epsilon$. Then the left-hand member of (10) will differ from

$$\int_h^H F(x) \, d\psi_1(x)$$

by not more than $\epsilon\, \psi_1(H)$. From the definition of $F(x)$ we have

$$\int_h^H F(x) \, d\psi_1(x) = \psi_1(x') + \epsilon',$$

where ϵ' is a non-negative quantity not exceeding $\psi_1(x' + \zeta) - \psi'(x')$. Hence by choosing ϵ and ζ sufficiently small we can make the first member of (10) to differ from $\psi_1(x')$ as little as we please, and similarly for its second member. Consequently we must have by (10) $\psi_1(x') = \psi_2(x')$. Application of this result to (8) makes $\psi_f(y) \equiv \psi_\phi(y)$ inasmuch as both of these functions possess right-handed continuity. Thus Haskins's theorem is established.

In the preceding paragraph the values of $\psi_1(x)$ and $\psi_2(x)$ were altered

only in their points of discontinuity. They are therefore identical excepting possibly at these points. This conclusion can be extended at once to two functions $\psi_1(x)$, $\psi_2(x)$ of limited variation. For if in (10) each function is expressed as the difference of two non-decreasing functions and the negative integral on each side of the equation is transposed to the other, the preceding result becomes applicable.

The identity of $\psi_1(x)$ and $\psi_2(x)$ except at points of discontinuity can also be deduced from work of Stieltjes. In his remarkable 1894 memoir* he supposed given the values of the constants

$$c_n = \int_0^\infty x^n \, d\psi(x) \qquad (n = 0, 1, 2, \cdots), \tag{11}$$

and considered $\psi(x)$ as a monotone, non-decreasing function to be determined, for which $\psi(0) = 0$. We have then his "Problem of Moments" for which he determines two cases, a determinate and an indeterminate one. To bring the integrals in (9) into the form (11), it will be necessary to change the origin when h is negative. For this purpose place $x = x' + h$ so that (9) becomes

$$\int_0^{H-h} (x' + h)^n \, d\bar{\psi}_1(x') = \int_0^{H-h} (x' + h)^n \, d\bar{\psi}_2(x'),$$

where $\bar{\psi}_1(x')$, $\bar{\psi}_2(x')$ are the transformed functions. On putting in succession $n = 0, 1, 2, \cdots$, we obtain

$$\int_0^{H-h} x'^n \, d\bar{\psi}_1(x') = \int_0^{H-h} x'^n \, d\bar{\psi}_2(x').$$

Consequently we may suppose $h = 0$ without restricting the problem.

In solving the problem of moments Stieltjes connects the formal expansion

$$\int_0^\infty \frac{d\psi(x)}{z + x} = \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots$$

with a corresponding continued fraction

$$\frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \cdots$$

in which a_n is positive. The constants c_n can not be arbitrarily chosen. One of the conditions to which they are subject is $c_{n+1}/c_n > c_n/c_{n-1}$, a condition which is also noted by Haskins. Supposing these conditions fulfilled, the necessary and sufficient condition for the existence of the determinate case is the divergence of $\sum a_n$; and the monotone function $\psi(x)$ is then completely determined by (11) except at a countable set of points which are points of discontinuity.

*Annales de la Faculté des Sciences de Toulouse, vol. 8, 1894.

Now the ratio c_{n+1}/c_n increases with n, and Stieltjes notes* that when it does not increase without limit, the numbers $b_n = 1/a_n a_{n+1}$ are limited. Consequently $\sum a_n$ is then divergent. If the interval of integration is finite, c_{n+1}/c_n can not increase without limit, for since $\psi(x)$ is monotone we have by the first mean value theorem

$$c_{n+1} \equiv \int_0^H x^{n+1}\, d\psi(x) = \xi \int_0^H x^n\, d\psi(x) = \xi c_n,$$

in which $0 < \xi \leqq H$. Hence the integrals (9) fall under the determinate case of (11), and the two functions $\psi_1(x)$, $\psi_2(x)$ are accordingly identical except at their points of discontinuity.

In conclusion, it may be remarked that the problem of moments presented under (3) may be reformulated so as to be embraced under (9). For this purpose put

$$F(x) = \int_a^x f(x)\,dx, \qquad \Phi(x) = \int_a^x \phi(x)\,dx.$$

Suppose first that $f(x) \geqq 0$ between a and b. If we then partition the interval (ab) (including 0 as a point of division in case it lies within the interval), and apply the mean value theorem for Lebesgue integrals, to each subinterval, we obtain

$$(12)\quad \int_a^b x^n f(x)\,dx = \lim_{m=\infty} \sum_{i=1}^{m} \xi_i^n \int_{x_i}^{x_{i+1}} f(x)\,dx = \lim \sum \xi_i^n\,[F(x_{i+1}) - F(x_i)],$$

where the ξ_i denote appropriately chosen points in the subintervals (x_i, x_{i+1}). As $F(x)$ is of limited variation, the last limit in (12) is a Stieltjes integral

$$\int_a^b x^n\,dF(x).$$

If $f(x)$ is not of constant sign in the interval, put $f(x) = f_1(x) - f_2(x)$, where $f_1(x) = 0$ when $f(x)$ is negative and $f_2(x) = 0$ when $f(x)$ is positive. By consideration of $f_1(x)$ and $f_2(x)$ separately we get finally the same Stieltjes integral. Hence our equations (3) may be replaced by the equations

$$\int_a^b x^n\,dF(x) = \int_a^b x^n\,d\Phi(x) \qquad (n = 0, 1, 2, \cdots).$$

As $F(x)$ and $\Phi(x)$ are of limited variation, they may each be expressed as the difference of two non-decreasing functions, so that by transposition we get an equation of form (9) with monotone integrands.

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* Loc. cit., p. 23.

POINT SETS AND ALLIED CREMONA GROUPS*

(PART III)

BY

ARTHUR B. COBLE†

INTRODUCTION

In Part I‡ of this series projectively distinct sets P_n^k of n points in S_k were mapped upon points of a space $\Sigma_{k(n-k-2)}$ and a certain Cremona group $G_{n!}$ in Σ was induced by permutation of the points of the set. For P_5^2 and P_6^2 these groups furnished an effective algebraic background for exhibiting solutions of the quintic and sextic equations. In Part II§ the $G_{n!}$ appeared as merely a subgroup of a more important group $G_{n,\,k}$ in $\Sigma_{k(n-k-2)}$ which also is defined by P_n^k. In particular the $G_{6!}$ in Σ_4 attached to P_6^2 is a subgroup of the $G_{6,\,2}$ in Σ_4, which has the order 51840 and is isomorphic with the group of the lines on a cubic surface.

It is the purpose of this Part III to show first that the lines of a given cubic surface can be determined rationally in terms of a solution of the form problem of $G_{6,\,2}$; and second that this solution can be obtained in terms of the solution of a form problem which arises in connection with the theta functions.‖ The presentation follows a line quite different from that suggested by Klein. A striking difference is that we make no actual or implied use of an equation of degree 27 or other resolvent equation. All of the operations required are effected within the domain of irrational invariants and covariants of the

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† This investigation has been carried out under the auspices of the Carnegie Institution of Washington, D. C.

‡ These Transactions, vol. 16 (1915).

§ These Transactions, vol. 17 (1916).

‖ That an equation of degree 27 for the lines of a cubic surface could be solved by hyperelliptic modular functions was first pointed out by Klein in a letter to Hermite, Journal de mathématiques, ser. 4, vol. 4 (1888), p. 169. His suggestions were elaborated by Witting, Mathematische Annalen, vol. 29 (1887), p. 167; by Maschke, ibid., vol. 33 (1889), p. 317; and by Burkhardt, *Grundzüge einer allgemeinen Systematik der hyperelliptischen Functionen I. Ordnung*, ibid., vol. 35 (1890), p. 198, vol. 38 (1891), p. 161, and vol. 41 (1893), p. 313. The latter articles are referred to as BI, BII, and BIII. Since Maschke and Burkhardt respectively have determined complete systems for the so-called "group of the Z's" and "group of the Y's" we shall indicate these groups and the associated form problems by attaching their names.

surface—a domain defined by the adjunction of all the lines considered as comitants of the surface.* Again the method of Klein is based primarily on the existence of the Maschke group and its derived group of line transformations while $G_{6,2}$ leads more naturally to the Burkhardt group.† We are thus enabled to dispense with a separation of the roots of the sextic underlying the theta functions of genus two. Finally by developing a covariant form for the normal hyperelliptic surface certain interesting geometric facts, pertaining to the collineation groups in question, are obtained upon which the solution of their form problems is based.‡

In § 1 the $G_{6,2}$, its generators, invariants, and form problem are discussed, while in § 2 it is shown that the adjunction of a solution of this form problem serves to determine the lines of a given cubic surface. In § 3 the simplest linear system of irrational invariants of the surface is set forth. Under permutation of the lines the members of this system are transformed under the operations of a correlation group which is built up on the Burkhardt collineation group. The form problem of $G_{6,2}$ is solved in § 4 by the adjunction of a solution of the Burkhardt form problem. The facts obtained in § 5 concerning the hyperelliptic surface serve in § 6 as a basis for the solution of the special§ Burkhardt form problem in terms of hyperelliptic modular functions. Finally in § 7 the general Burkhardt form problem is solved in terms of the special problem. The comparison of this determination of the lines of a cubic surface with the solution of a quintic equation detailed in § 8 reveals a remarkable analogy between these two problems—an analogy which furnishes perhaps the best evidence of the value of the methods here employed.

1. The extended group $G_{6,2}$

For a given ordered set of 6 points in a plane we choose as in Part I the reference scheme and factors of proportionality so that the coördinates of the points take the form

$$(1)\qquad \begin{array}{ll} (1)\ 1,\ 0,\ 0, & (4)\ 1,\ 1,\ 1, \\ (2)\ 0,\ 1,\ 0, & (5)\ x,\ y,\ u, \\ (3)\ 0,\ 0,\ 1, & (6)\ z,\ t,\ u. \end{array}$$

The P_6^2 is then represented by the point P in Σ_4 with coördinates x, y, z, t, u.

* This removes the objection—noted by Burkhardt (BIII, p. 342, 3)—that while the lines are in rational relation they are not in covariant relation to the roots of such an equation. Equations whose roots are irrational invariants of the surface can be formed but the degree of these invariants is too high for practical discussion.

† Since no solution of the Burkhardt form problem has been given hitherto this is supplied below.

‡ The results here obtained indicate some errors in the formulas of Burkhardt; cf. footnotes §§ 4, 5, 6.

§ I. e., the Burkhardt form problem for which $J_4 = 0$.

As the points of P_6^2 are permuted and the points of the re-ordered set transformed again into the form (1) the point P is transformed into its conjugate positions under $G_{6!}$. A set of generators $I_{12}, \cdots, I_{56}$ of $G_{6!}$ is given in (2) below where I_{ik} is the involutory operation of $G_{6!}$ which arises from the interchange of the points p_i and p_k of P_6^2. A quadratic transformation I_{123} with fundamental points (F-points) at p_1, p_2, p_3 and corresponding inverse F-points at p_1', p_2', p_3' transforms p_4, p_5, p_6 into p_4', p_5', p_6'. Then by definition the sets P_6^2 and $P_6'^2$ are *congruent* under I_{123}. If the set $P_6'^2$ be projected into the form (1) the coördinates of its map P' in Σ_4 in terms of those of P are (II, § 3) as given in the table:

(2)

	I_{12}	I_{23}	I_{34}	I_{45}	I_{56}	I_{123}	I
$x' =$	y	xt	$u - x$	uy	z	$yztu$	$x + \alpha$
$y' =$	x	ut	$u - y$	ux	t	$xztu$	$y + \alpha$
$z' =$	t	zy	$u - z$	zy	x	$xytu$	$z + \alpha$
$t' =$	z	uy	$u - t$	tx	y	$xyzu$	$t + \alpha$
$u' =$	u	yt	u	xy	u	$xyzt$	βu

$$\alpha = \frac{zy(x+t-u) - xt(y+z-u)}{xt - yz - u(x+t-y-z)}, \qquad \beta = 1 + \alpha\frac{x+t-y-z}{xt-yz}.$$

Here I (the element B of I, § 10) is the involutory transformation determined by associated sets P_6^2 and $P_6'^2$.

We have seen (II, § 3, p. 358) that the $G_{6!}$ generated by $I_{12}, \cdots, I_{56}$ together with the I_{123} generate a group $G_{6,2}$ in Σ_4 of order 51840; and further that the 15 involutions I_{ik}, the 20 involutions I_{ijk}, and the involution I constitute a set of 36 conjugate generators of $G_{6,2}$. Those elements of $G_{6,2}$ which can be expressed as a product of an even number of generators lie in the invariant subgroup $\Gamma_{6,2}$ (II, § 7 (51)) of order 25920. To a set of 51840 conjugate points under $G_{6,2}$ in Σ_4 there corresponds in the plane a set of 51840 projectively distinct ordered P_6^2's such that any one of the P_6^2's is congruent in some order under ternary Cremona transformation to any other of the P_6^2's. Disregarding the 6! possible orders we have then just 72 projectively distinct P_6^2's. The 72 P_6^2's divide under I into 36 pairs of associated P_6^2's. The original P_6^2 and its associated P_6^2 determine the group $G_{6!\cdot 2}$ of I, § 10. The property of congruent P_6^2's most important for our purpose is

(3) *If the system of cubic curves on P_6^2 can be transformed by ternary Cremona transformation into the system of cubic curves on $P_6'^2$ then the maps of P_6^2 and $P_6'^2$ in Σ_4 are conjugate points under $G_{6,2}$.*

According to I, § 4, we can choose six cubic curves $a, \cdots, f$ on P_6^2 such that

$$(4) \qquad \begin{aligned} a + \cdots + f &\equiv 0, \\ \bar{a}a + \cdots + \bar{f}f &\equiv 0, \\ \bar{a} + \cdots + \bar{f} &= 0. \end{aligned}$$

Explicit expressions for $a, \cdots, f, \bar{a}, \cdots, \bar{f}$ in terms of the coördinates of P_6^2 are given there. The plane is thus mapped upon a cubic surface $C^{(3)}$ whose equation is

$$(5) \qquad C^{(3)} \equiv a^3 + \cdots + f^3 = 0.$$

The equations of the 45 tritangent planes (and thereby also the equations of the 27 lines) of $C^{(3)}$ also are explicitly given in terms of P_6^2. The above hexahedral form of $C^{(3)}$ is associated with a double six of lines on $C^{(3)}$ and precisely that double six mapped from the six points p_i of P_6^2 and the six conics on points other than p_i $(i = 1, \cdots, 6)$.

The invariants of the hexahedral $C^{(3)}$ are symmetric polynomials in $\bar{a}, \cdots, \bar{f}$ which are unaltered to within a factor when $\bar{a}, \cdots, \bar{f}$ are replaced by the $\bar{a}', \cdots, \bar{f}'$ attached to any other hexahedral form of $C^{(3)}$. The complete system of a cubic surface in S_3 according to Salmon and Clebsch* contains invariants $i_8, i_{16}, i_{24}, i_{32}$, and i_{100} of the degrees indicated and of weights 6, 12, 18, 24, 30, and 75 respectively. The hexahedral surface appears as a section in S_5 (variables $a, \cdots, f$) of a cubic spread with numerical coördinates. The Clebsch transference principle states that the invariants of the section are obtained by bordering the symbolic determinants of the invariant in S_3 by the coördinates of the spaces of section and by applying the determinants to the spread in S_5. Since the coefficients of the spread in S_5 are merely numerical each unit of weight contributes a unit to the degree in $\bar{a}, \cdots, \bar{f}$, and the invariants of the hexahedral form are $I_6, I_{12}, I_{18}, I_{24}, I_{30}$, and I_{75} of the degrees indicated in $\bar{a}, \cdots, \bar{f}$. When $\bar{a}, \cdots, \bar{f}$ are expressed in terms of the x, y, z, t, u of P_6^2 or of the point P in Σ_4 the process of expressing an invariant I in terms of the $\bar{a}', \cdots, \bar{f}'$ corresponds to the process of replacing P_6^2 by a congruent P'^2_6, or in Σ_4 to the process of passing to a conjugate point P' under $G_{6,2}$. Since the relation of I to $C^{(3)}$ is unaltered by this operation, the corresponding expression for I in terms of x, y, z, t, u must be unaltered to within a factor by the operations of $G_{6,2}$. Hence to the invariants I of weight w of $C^{(3)}$ there correspond invariant spreads I of $G_{6,2}$ in Σ_4 determined to within a power λ^w of an indeterminate factor of proportionality λ. That but one such factor λ can occur is due to the fact that linear systems of invariants can be formed as soon as two invariants appear. Here the invariant I_{75} may be an exception since it cannot lie in a linear system determined by itself and earlier invariants because of its odd degree. This ambiguity in the factor λ can however extend only to the sign since I_{75}^2 is no longer exceptional. Conversely every invariant I of $G_{6,2}$ must arise in this way from some invariant of $C^{(3)}$. By an invariant I of $G_{6,2}$ we mean a form in x, y, z, t, u which is reproduced to within a factor by any operation of $G_{6,2}$. For the coördinates x, y, z, t, u of P are themselves irrational invariants of

* Cf. Pascal; *Repertorium* I, p. 342 (Teubner, 1900).

$C^{(3)}$* in the projective sense which are rational in the domain obtained by the separation of the lines. The given form is therefore a projective invariant in this domain and, being invariant under the operations of $G_{6,2}$, it must be independent of admissible permutations of the lines and therefore must be rational in the coefficients of $C^{(3)}$. According to I, p. 196, the $\bar{a}, \cdots, \bar{f}$ are of degree two in x, y, z, t, u and we shall find† that for $w = 6k$ the factor u^{2k} can be removed from each invariant of weight w, whence

(6) *To the complete system of invariants* $i_8, \cdots, i_{32}, i_{40}, i_{100}$ *of the general* $C^{(3)}$ *there corresponds the complete system* $I_6, \cdots, I_{24}, I_{30}, I_{75}$ *of the hexahedral* $C^{(3)}$, *and the complete system* $I_{5\cdot 2}, \cdots, I_{5\cdot 8}, I_{5\cdot 10}, I_{5\cdot 25}$ *of spreads of the orders indicated invariant under* $G_{6,2}$ *in* Σ_4.

The equation problem determined by the $G_{6,2}$ can now be formulated as follows:

(7) *Given numerical values of* $I_{5\cdot 2}, I_{5\cdot 4}, \cdots, I_{5\cdot 10}$ *to calculate the ratios of the coördinates* x, y, z, t, u *of a point* P *in* Σ_4 *for which the given spreads can take the assigned values.*

From one solution of this problem there can be obtained by the operations of $G_{6,2}$ a set of 51840 solutions. It is not easy to prove directly that no other solutions exist, since these spreads have common manifolds determined by the reference basis in Σ. The fact can be inferred later from the number of solutions of the Burkhardt form problem.

It had been noted by Klein (loc. cit.) that the adjunction of the square root of the discriminant Δ of $C^{(3)}$ reduced the order of the group of the lines to 25920. Evidently Δ is the most convenient fourth invariant and we shall suppose hereafter that i_{32}, I_{24}, or $I_{5\cdot 8}$, as the case may be, is Δ. The $\sqrt{\Delta}$ can be calculated explicitly in terms of the $\bar{a}, \cdots, \bar{f}$ without much trouble. The surface when mapped by means of P_6^2 has a double point when either six points are on a conic, or three points are on a line, or two points coincide in some direction. Thus Δ has $1 + 20 + 15 = 36$ irrational factors which will be indicated respectively by $\delta, \delta_{ijk}, \delta_{ij}$. Recalling the notation of I, pp. 170–3, we have $\delta = d_2$ and $\delta_{123}\,\delta_{456} = (\bar{a} + \bar{b} + \bar{c}) = -(\bar{d} + \bar{e} + \bar{f})$, etc. In order that a rational expression in $\bar{a}, \cdots, \bar{f}$ may be obtained we must use $d_2^2 = a_2^2 - 4a_4$ and therefore must use the squares of the factors δ_{ijk} as well. This raises the product of δ^2 and the ten squares δ_{ijk}^2 to the proper degree, 24, of Δ in $\bar{a}, \cdots, \bar{f}$ so that the factors δ_{ij} do not occur explicitly. This is to be expected since coincidence in a given direction cannot be expressed by a single condition on the coördinates of P_6^2. A definite sign can be given to $\sqrt{\Delta}$ by the assumption

$$\sqrt{\Delta} = d_2 \prod_{10} (\bar{a} + i + j), \tag{8}$$

* E. g., if $x = 0$, points p_2, p_3, p_5 are collinear and $C^{(3)}$ has a node.

† Cf. § 3.

where the sign of d_2 is defined in I, § 4 (47), and where i, j run over the 10 pairs drawn from $\bar{b}, \cdots, \bar{f}$. The product $\prod$ can be conveniently evaluated by symmetrizing successively for $3, 4, \cdots, 6$ letters. If in terms of $a_2, \cdots, a_6$, the elementary symmetric functions of $\bar{a}, \cdots, \bar{f}$, we set

$$
\begin{aligned}
q_4 &= a_2^2 - 4a_4, \qquad q_5 = a_2 a_3 - 2a_5, \\
I_6{}^* &= 3a_3^2 - 4a_2 q_4 - 12a_6,
\end{aligned} \tag{9}
$$

then $\sqrt{\Delta}$ takes the form

$$
\sqrt{\Delta} = d_2(q_4 I_6 + 4a_2 q_4^2 - 3q_5^2). \tag{10}
$$

We have seen (I, p. 196) that d_2 changes sign under the involution I_{12} while $\bar{a}, \cdots, \bar{f}$ and therefore $a_2, \cdots, a_6, q_4, q_5, I_6$ do not; whence $\sqrt{\Delta}$ changes sign under the operations of $G_{6,2}$ not contained in $\Gamma_{6,2}$ and is invariant under $\Gamma_{6,2}$.

The skew invariant, i_{100} or I_{75}, has the same behavior under $G_{6,2}$ and $\Gamma_{6,2}$ as $\sqrt{\Delta}$. This invariant has 45 irrational factors. For if we recall from I, p. 197, that $\bar{a} - \bar{d} = 0$ is the condition that the lines $\overline{12}, \overline{34}, \overline{56}$ in S_2 meet in a point we find 15 similar irrational invariants whose product is there denoted by $\sqrt{d}$. If then we carry out on $\bar{a} - \bar{d} = 0$ the involution I_{123} it must be transformed into the condition that the conic on p_1, p_2, p_5, p_6 is touched at p_3 by the line $\overline{34}$. This condition is

$$
j_{1256,\,3} \equiv \begin{vmatrix} \overline{126}\ \overline{356} & \overline{136}\ \overline{256} \\ \overline{123}\ \overline{354} & \overline{134}\ \overline{253} \end{vmatrix} = 0. \tag{11}
$$

Thus $j_{1256,\,3}$ is of degree $2, 2, 3, 1, 2, 2$ in the points in order. The product of the 30 irrational invariants of this type is of degree 60 in each of the six points; let us call it f_{60}. It is easy to verify that, while $j_{1256,\,3}$ is changed in sign by I_{12}, yet f_{60} is unaltered. Since f_{60} does not contain the factor d_2 it must (cf. I (49)) be a rational integral function of $a_2, \cdots, a_6$. Hence

$$
I_{75} = \sqrt{d} \cdot f_{60} \tag{12}
$$

is changed in sign along with $\sqrt{d}$ by the involution I_{12} and therefore is an exception to the rule that invariants of $C^{(3)}$ are invariants under $G_{6,2}$.

The above identification of $\sqrt{d} \cdot f_{60}$ with I_{75} which is indicated by their degrees can be substantiated by their geometric interpretation. The vanishing of I_{75} is the condition that $C^{(3)}$ be unaltered by an involutory collineation. On the other hand if $\bar{a} - \bar{d} = 0$ then (cf. I (16)) P_6^2 is self-associated in the order $(12)(34)(56)$, the net of cubics on P_6^2 is transformed into itself by the corresponding Cremona involution of order 5, and the cubic surface admits an involutory collineation.

(13) *A complete system of invariants for $\Gamma_{6,2}$ consists of $I_6, I_{12}, \sqrt{\Delta}, I_{18}, I_{30}$, and I_{75}.*

* We shall verify later (§ 3 (38)) that this is the first invariant of $C^{(3)}$.

The degrees of all the invariants of $\Gamma_{6,\,2}$ which we shall have occasion to use hereafter are such that I_{75} cannot occur in their expressions. The form problem for the group $\Gamma_{6,\,2}$ reads as follows:

(14) *Given the numerical values of $I_{5\cdot 2}$, $I_{5\cdot 4}$, $\sqrt{\Delta}$, $I_{5\cdot 6}$, and $I_{5\cdot 10}$ to calculate the ratios of the coördinates of a point P in Σ_4 for which these spreads can take the assigned values.*

Clearly the adjunction of $\sqrt{\Delta}$ suffices to reduce the group $G_{6,\,2}$ to the group $\Gamma_{6,\,2}$. For the linear system $\lambda I_6^2 + \mu I_{12} + \nu\sqrt{\Delta}$ is invariant under $\Gamma_{6,\,2}$ but not under $G_{6,\,2}$. Hence if the form problem of $G_{6,\,2}$ has 51840 solutions, that of $\Gamma_{6,\,2}$ has 25920 solutions, and vice versa.

2. Determination of the lines of a cubic surface in terms of the form problem of $G_{6,\,2}$

If $(cx)^3$ is a general quaternary cubic form the cubic surface $C^{(3)}$ is $(cx)^3 = 0$. We seek the 27 sets of six line coördinates π_{ik} determined by any pair of planes on each of the 27 lines of $C^{(3)}$. Such pairs can be selected from the 45 tritangent planes of $C^{(3)}$ so that we shall merely need expressions $(t_i\, x) = 0$ $(i = 1, \cdots, 45)$ for these planes.

We shall assume first that for the given form $(cx)^3$ a series of covariant processes has been outlined, following some one of the known complete systems of the cubic surface, which will furnish definite values for the invariants i_8, i_{16}, i_{24}, i_{32}, and i_{40} of $C^{(3)}$ and definite expressions, $(l_{11}\, x)$, $(l_{19}\, x)$, $(l_{27}\, x)$, and $(l_{43}\, x)$ for the linear covariants of $C^{(3)}$.

In any one of 36 ways the given form $(cx)^3$ can be expressed in Cremona's hexahedral form (4). We shall assume second that through the use of the Clebsch transference principle which does not affect covariant relations, the same series of covariant processes has been carried out on the Cremona form, and that there has been obtained explicit expressions in terms of $\bar{a}, \cdots, \bar{f}$ of the corresponding invariants and linear covariants, viz: I_6, I_{12}, I_{18}, I_{24}, I_{30}, and $(L_8\, a)$, $(L_{14}\, a)$, $(L_{20}\, a)$, $(L_{32}\, a)$ of the degrees in $\bar{a}, \cdots, \bar{f}$ indicated by the subscripts. The two investigations here assumed have of course a wider range of application than we shall need. The first is in quite satisfactory shape. The second has been begun* and if completed would afford a method for settling all questions concerning the relation of the lines and tritangent planes of $C^{(3)}$ to the covariant system of $C^{(3)}$.

If the coefficients of $(cx)^3$ and $\bar{a}, \cdots, \bar{f}$ be known then the linear transformation

$$(15)\qquad \begin{aligned} (l_{11}\, x) &= (L_8\, a), & (l_{19}\, x) &= (L_{14}\, a), \\ (l_{27}\, x) &= (L_{20}\, a), & (l_{43}\, x) &= (L_{32}\, a), \\ 0 &= (\bar{a}a), & 0 &= (a), \end{aligned}$$

* Sousley, American Journal of Mathematics, vol. 38 (1916).

furnishes the typical representation of the given cubic surface in the hexahedral form. The determinant of the four linear forms in x of (15) is i_{100}; of the six linear forms in $\bar{a}, \cdots, \bar{f}$ is I_{75}.

The processes involved in the determination of the 45 tritangent planes of the given cubic surface $C^{(3)}$ can now be outlined as follows:

1°. For the given $C^{(3)} \equiv (cx)^3 = 0$, the invariants $i_8, \cdots, i_{40}$ and the linear covariants $(l_{11}x), \cdots, (l_{43}x)$ are calculated.

2°. The values of $i_8, \cdots, i_{40}$ furnish the values of the known quantities $I_6, \cdots, I_{30}$ of the equation problem of $G_{6,2}$. We assume that this problem has been solved and that therefore the coördinates x, y, z, t, u of a point P in Σ_4 are known.

3°. These coördinates substituted in (1) furnish a set P_6^2 by means of which the given $C^{(3)}$ can be mapped upon S_2. The values of $\bar{a}, \cdots, \bar{f}$ and d_2 in terms of these coördinates are given in I (35), (47).

4°. From the values of $\bar{a}, \cdots, \bar{f}$ and d_2 the linear covariants $(L_8 a), \cdots, (L_{32} a)$ of (15), and the 45 tritangent planes as given in I (47) are expressed linearly in terms of $a, \cdots, f$.

5°. From equations (15) the values of $a, \cdots, f$ as linear functions of x are obtained.

6°. These values of $a, \cdots, f$ set in the equations of the 45 tritangent planes furnish the equations of the tritangent planes of the given surface $(cx)^3 = 0$ in terms of the given variables x.

The above procedure is effective whenever the equations (15) are such that step 5° can be carried out. We have therefore a case of exception when $i_{100} = I_{75} = 0$. In this case however the Galois group of the problem is reduced at least to a desmic group of order $2 \cdot 576$ and the lines of $C^{(3)}$ can be expressed by means of radicals alone.

Since any solution of the equation problem of $\Gamma_{6,2}$ is equally well a solution of that of $G_{6,2}$, all the operations outlined above can be carried out after the adjunction of $\sqrt{\Delta}$ has reduced the group of the problem to $\Gamma_{6,2}$ of order 25920. We are now enabled to dispense with the originally given surface $C^{(3)}$ and to consider further only those processes which are necessary for the solution of the equation problem of $\Gamma_{6,2}$.

3. Irrational invariants of $C^{(3)}$

In this paragraph certain systems of irrational invariants of $C^{(3)}$ are built up from the irrational factors of its discriminant. They correspond to the separation of the lines of the surface into double sixes, into the so-called "complexes" of lines, and into the triads of lines in a tritangent plane. Of these three sorts the first do not give rise to irrational invariants of P_6^2 as defined in I, § 3, for their expressions in terms of the coördinates of P_6^2 are not

homogeneous and of the same degree in the coördinates of each point. But the last two sorts are irrational invariants of P_6^2 and they lie in the simplest linear system of spreads invariant under the extended group $G_{6,\,2}$ of P_6^2. The operations of $G_{6,\,2}$ when carried out on this linear system give rise to a collineation group of order 51840 which we shall identify with a group of correlations in S_4 which has the Burkhardt group for its collineation subgroup.

Let us denote the lines of $C^{(3)}$ by $l_1, \cdots, l_6$; $m_1, \cdots, m_6$; $l_{12}, \cdots, l_{56}$ according as they correspond in the plane of P_6^2 to respectively the directions about the points $p_1, \cdots, p_6$; the conics on the six points other than $p_1, \cdots, p_6$; and the lines

$$(12x) = 0, \qquad \cdots, \qquad (56x) = 0.$$

In the above notation a double six is isolated and the 36 double sixes divide into three sets

$$D \equiv \left\{ \begin{matrix} l_1, & l_2, & \cdots, & l_6 \\ m_1, & m_2, & \cdots, & m_6 \end{matrix} \right\},$$

$$D_{lmn} \equiv \left\{ \begin{matrix} l_{jk}, & l_{ik}, & l_{ij}, & l_l, & l_m, & l_n \\ m_i, & m_j, & m_k, & l_{mn}, & l_{ln}, & l_{lm} \end{matrix} \right\},$$

$$D_{ij} \equiv \left\{ \begin{matrix} l_i, & m_i, & l_{jk}, & l_{jl}, & l_{jm}, & l_{jn} \\ l_j, & m_j, & l_{ik}, & l_{il}, & l_{im}, & l_{in} \end{matrix} \right\},$$

$$(i, j, k, \cdots, n = 1, 2, \cdots, 6).$$

In the plane, D_{lmn} arises from D by using the quadratic transformation A_{ijk}; while D_{ij} arises from D by using the cubic transformation with double F-point at p_j and simple F-points at $p_k, \cdots, p_n$. The double six D has 6 lines in common with each of the 20 double sixes D_{lmn} and 4 lines in common with each of the 15 double sixes D_{ij}. Two double sixes form an *azygetic* or *syzygetic duad* according as they have 6 or 4 lines in common. The typical azygetic duads are:

$$D, D_{ijk}; \quad D_{ijk}, D_{lmn}; \quad D_{ijk}, D_{ijl}; \quad D_{ijk}, D_{il}; \quad D_{ij}, D_{ik};$$

the typical syzygetic duads are:

$$D, D_{ij}; \quad D_{ijk}, D_{ilm}; \quad D_{ijk}, D_{lm}; \quad D_{ijk}, D_{ij}; \quad D_{ij}, D_{kl}.^*$$

* The terms azygetic and syzygetic are taken from the finite geometry mod 2 ($p = 3$) of the theta functions (cf. II, p. 357). If of the 28 O quadrics one, say Q_{07}, be isolated then the 36 points P_{07}, $P_{0ijk} = P_{lmn7}$, P_{ij} which are not on this quadric can be associated with the double sixes while the 27 points P_{0i}, P_{7i}, and P_{ij07} which are on the quadric can be associated with the 27 lines of $C^{(3)}$. Through each point outside the quadric there pass 6 secants of the quadric. These are ordinary lines of the null system determined by the quadric. Two points are syzygetic or azygetic according as they do or do not lie on a null line. Thus each outside point (or double six) determines 6 pairs of points on Q_{07} (line pair of $C^{(3)}$) and these 6 pairs lie in two sets of 6 for each point is azygetic with all of its own set and one of the other set. Any two outside points may be syzygetic or azygetic. In the first case their join touches the quadric at a point; in the second case their join is skew to the quadric and the three points on it correspond to an azygetic triad of double sixes.

Two azygetic double sixes determine a third such that any two of the three are azygetic. There are 120 such *azygetic triads* whose typical forms are:

$$D\,,\ D_{ijk}\,,\ D_{lmn};\qquad D_{ijk}\,,\ D_{ijl}\,,\ D_{kl};\qquad D_{ij}\,,\ D_{ik}\,,\ D_{jk}\,.$$

No line is common to the three double sixes of an azygetic triad so that the triad contains 18 lines each appearing in two double sixes and omits 9 lines.

The double sixes which are syzygetic with the members of an azygetic triad lie in two other azygetic triads such that double sixes from any two of the triads are syzygetic. Such a symmetrical set of three azygetic triads is a *complex*. The 9 lines omitted from each triad of a complex make up the 27 lines. There are 40 such complexes; 10 of the form

$$\Gamma_{ijk,\,lmn} \equiv D\,,\ D_{ijk}\,,\ D_{lmn};\quad D_{ij}\,,\ D_{ik}\,,\ D_{jk};\quad D_{lm}\,,\ D_{ln}\,,\ D_{mn};$$

and 30 of the form

$$\Gamma_{ij,\,kl,\,mn} \equiv D_{ij}\,,\ D_{ikl}\,,\ D_{jkl};\quad D_{kl}\,,\ D_{kmn}\,,\ D_{lmn};\quad D_{mn}\,,\ D_{mij}\,,\ D_{nij}\,.$$

The last type depends upon the separation of $i, \cdots, n$ into three pairs and upon the cyclic arrangement of the pairs.*

We have noted in § 1 that there are 36 particular types of sets P_6^2 for which $C^{(3)}$ has a double point, each type being associated with a double six. The six lines of half the isolated double six map into the six lines of $C^{(3)}$ on the double point and the six lines of the other half coincide and map into directions at the double point. The 36 corresponding irrational factors of the discriminant Δ of $C^{(3)}$ have been denoted by δ, δ_{lmn}, δ_{ij}. The vanishing of each of these factors can be expressed by explicit conditions on the coördinates of P_6^2 except for the factors of type δ_{ij} which indicate a coincidence of the points p_i and p_j.† We shall now set forth a set of 40 irrational invariants of $C^{(3)}$ which correspond to the 40 complexes defined above.

If we consider the product

$$\gamma_{123,\,456} = d_2 \cdot (123)(456) \tag{16}$$

we see that it is of degree 3 in the coördinates of each point of P_6^2; that it vanishes at least once for each coincidence δ_{ij}; that it vanishes twice for the coincidences δ_{12}, δ_{13}, δ_{23}; δ_{45}, δ_{46}, δ_{56}; and that it vanishes with the factors δ, δ_{123}, and δ_{456} of Δ. It corresponds therefore to the product of the 9 discriminant factors associated with the 9 double sixes in the complex $\Gamma_{123,\,456}$. In order to derive the remaining nine "complex invariants" of this type

* All of these configurations of lines are well known; cf. Pascal, loc. cit., II, pp. 284–90.

† If P_6^2 is taken in the canonical form (1) the same exception applies to δ_{234}, δ_{134}, δ_{124}, and δ_{123}. A closer study of the invariants of $G_{6,\,2}$ in Σ_4 would reveal the singular manifolds which correspond to these factors of Δ as well as to factors of type δ_{ij}. This information however is not necessary for our purposes.

by permutation of the points* we shall complete the definition (16) by the further requirements:

$$\begin{aligned}(17)\qquad \gamma_{ijk,\,lmn} &= \gamma_{jik,\,lmn} = \gamma_{jki,\,lmn}\\ &= \gamma_{ijk,\,mln} = \gamma_{ijk,\,mnl} = -\gamma_{lmn,\,ijk}.\end{aligned}$$

Then the ten complex invariants of the first type are

TABLE I

$$\begin{aligned}
\gamma_{123,\,456} &= d_2(123)(456), & \gamma_{125,\,463} &= d_2(125)(463),\\
\gamma_{134,\,562} &= d_2(134)(562), & \gamma_{136,\,524} &= d_2(136)(524),\\
\gamma_{145,\,623} &= d_2(145)(623), & \gamma_{142,\,635} &= d_2(142)(635),\\
\gamma_{156,\,234} &= d_2(156)(234), & \gamma_{153,\,246} &= d_2(153)(246),\\
\gamma_{162,\,345} &= d_2(162)(345), & \gamma_{164,\,352} &= d_2(164)(352).
\end{aligned}$$

From equations I (34) we get at once the values of the $\gamma_{ijk,\,lmn}$ in terms of $d_2, \bar{a}, \cdots, \bar{f}$.

Consider again the product

$$(18)\qquad \gamma_{ij,\,kl,\,mn} = (ikl)(jkl)(kmn)(lmn)(mij)(nij).$$

It also is of degree 3 in the coördinates of each point of P_6^2; it vanishes at least once for every coincidence; it vanishes twice for the coincidences δ_{ij}, δ_{kl}, and δ_{mn}; it vanishes along with the six factors δ_{ikl}, etc., of Δ; and it therefore corresponds as above to the complex $\Gamma_{ij,\,kl,\,mn}$. Its definition is completed by the further requirements (which accord with permutations of the points of P_6^2):

$$(19)\qquad \gamma_{ij,\,kl,\,mn} = \gamma_{kl,\,mn,\,ij} = \gamma_{ji,\,kl,\,mn} \neq \gamma_{ij,\,mn,\,kl}.$$

Hence there are 30 complex invariants of this type which occur in 15 pairs.

In order to get explicit expressions for them in terms of $d_2, \bar{a}, \cdots, \bar{f}$ consider the particular pair $\gamma_{12,\,34,\,56}$ and $\gamma_{12,\,56,\,34}$. If in these we substitute respectively for $(134)(356)$ and $(256)(412)$ their values $(315)(364) + (316)(345)$ and $(264)(215) + (245)(216)$ we find that

$$\begin{aligned}\gamma_{12,\,34,\,56} &= (364)(512)\cdot(234)(456)(612)(315)\\ &\qquad + (345)(612)\cdot(234)(456)(512)(316),\\ \gamma_{12,\,56,\,34} &= (364)(512)\cdot(156)(312)(534)(264)\\ &\qquad + (345)(612)\cdot(156)(132)(634)(245).\end{aligned}$$

* We use here the parallel generating permutations of I (28), viz.:

$$(12),\ (23456);\ (\bar{a}\bar{d})(\bar{b}\bar{e})(\bar{c}\bar{f}),\ (\bar{a}\bar{d}\bar{b}\bar{f}\bar{e}),$$

an odd permutation requiring also a change of sign in d_2.

From I (46) and (47) we have*

$$(234)(456)(612)(315) \begin{matrix} + \\ - \end{matrix} (156)(312)(534)(264) = \begin{matrix} \overline{de} \\ -\,d_2, \end{matrix}$$

$$(234)(456)(512)(316) \begin{matrix} + \\ - \end{matrix} (156)(132)(634)(245) = \begin{matrix} -\,\overline{ac} \\ -\,d_2. \end{matrix}$$

$$\text{(20)} \qquad \therefore\ \gamma_{12,\,34,\,56} - \gamma_{12,\,56,\,34} = -\,d_2[(364)(512) + (345)(612)] = d_2(\bar{d} - \bar{a}).$$

$$\begin{aligned} \gamma_{12,\,34,\,56} + \gamma_{12,\,56,\,34} &= (\bar{a} + \bar{c} + \bar{e})\overline{de} + (\bar{d} + \bar{c} + \bar{e})\overline{ac} \\ &= (\bar{a} + \bar{c} + \bar{e})(a_2 + 2\bar{d}^2 + 2\bar{e}^2 + 2\bar{d}\bar{e}) \\ &\quad + (\bar{d} + \bar{c} + \bar{e})(a_2 + 2\bar{a}^2 + 2\bar{c}^2 + 2\bar{a}\bar{c}) \\ &= -\,a_2(\bar{a} + \bar{d}) - 2(\bar{a}^3 + \bar{d}^3) + 2[a_2\sigma_1 + \Sigma\bar{a}^3 + \Sigma\bar{a}^2\bar{d} + \sigma_3] \\ &= -\,a_2(\bar{a} + \bar{d}) - 2(\bar{a}^3 + \bar{d}^3) + 2[a_2\sigma_1 + \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3], \end{aligned}$$

where the Σ's refer to symmetric functions and the σ's to elementary symmetric functions of $\bar{a}$, $\bar{c}$, $\bar{d}$, $\bar{e}$. But from

$$\bar{b} + \bar{f} = -\,\sigma_1, \qquad \bar{b}\bar{f} + (\bar{b} + \bar{f})\sigma_1 = a_2 - \sigma_2,$$
$$\bar{b}\bar{f}\sigma_1 + (\bar{b} + \bar{f})\sigma_2 = a_3 - \sigma_3$$

we get by eliminating $\bar{b} + \bar{f}$ and $\bar{b}\bar{f}$ the relation

$$a_2\sigma_1 + \sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3 - a_3 = 0.$$

$$\therefore\ \gamma_{12,\,34,\,56} + \gamma_{12,\,56,\,34} = 2a_3 - a_2(\bar{a} + \bar{d}) - 2(\bar{a}^3 + \bar{d}^3).$$

If now we introduce the new set of six quantities $\bar{\alpha}, \cdots, \bar{\zeta}$ defined by

$$\text{(21)} \qquad \begin{aligned} \bar{\alpha} &= a_3 - a_2\bar{a} - 2\bar{a}^3, \\ &\cdots\cdots\cdots \\ \bar{\zeta} &= a_3 - a_2\bar{f} - 2\bar{f}^3, \end{aligned}$$

which satisfy the relation

$$\text{(22)} \qquad \bar{\alpha} + \bar{\beta} + \bar{\gamma} + \bar{\delta} + \bar{\epsilon} + \bar{\zeta} = 0,$$

then

$$\gamma_{12,\,34,\,56} + \gamma_{12,\,56,\,34} = \bar{\alpha} + \bar{\delta}.\dagger$$

* The formulæ used here are derived from those quoted by proper permutation.

† These six values which serve so well for the expression of the complex invariants have another interesting contact. If $\bar{a}, \cdots, \bar{f}$ are the roots of a sextic then (21) is the Tschirnhausen transformation to a sextic with roots $\bar{\alpha}, \cdots, \bar{\zeta}$. If $d_2 = 0$, i. e., if $a_2^2 - 4a_4 = 0$, the first sextic is the sextic of Maschke and the transformed sextic is that of Joubert. For it is easy to verify that $\Sigma\bar{\alpha}^3 = 9(a_2^2 - 4a_4)(a_2 a_3 - 2a_5) = q_4 q_5$. From this point of view the transformation (21) has been used in an earlier paper—Coble, *An application of Moore's cross-ratio group*, etc.; these Transactions, vol. 12 (1911), p. 323, (32); cited hereafter as C1.

Combining this with (20) we get values of $\gamma_{12,\,34,\,56}$ and $\gamma_{12,\,56,\,34}$ and by permutation obtain the values of the 30 complex invariants of the second type. These are collected in

TABLE II*

$$2\gamma_{12,\,34,\,56} = \bar{\alpha} + \bar{\delta} - d_2(\bar{a} - \bar{d}),$$
$$2\gamma_{12,\,56,\,34} = \bar{\alpha} + \bar{\delta} + d_2(\bar{a} - \bar{d}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot$$

The 40 complex invariants are connected by many relations. We see at once that the 10 invariants in Table I are connected by 15 relations of the type:

1°. $$\gamma_{123,\,456} + \gamma_{124,\,356} + \gamma_{134,\,256} + \gamma_{234,\,156} = 0.$$

The complexes which correspond to the four terms are those which contain the syzygetic duad of double sixes D, D_{56}. Corresponding therefore to the four other types of syzygetic duads listed above we have respectively four other types of relations:

2°. $$\gamma_{23,\,15,\,64} + \gamma_{12,\,35,\,64} - \gamma_{42,\,65,\,13} - \gamma_{62,\,45,\,13} = 0,$$

3°. $$\gamma_{456,\,123} + \gamma_{45,\,61,\,23} + \gamma_{45,\,62,\,31} + \gamma_{45,\,63,\,12} = 0,$$

4°. $$\gamma_{123,\,456} + \gamma_{23,\,14,\,56} + \gamma_{31,\,24,\,56} + \gamma_{12,\,34,\,56} = 0,$$

5°. $$\gamma_{12,\,34,\,56} - \gamma_{12,\,56,\,34} + \gamma_{156,\,234} - \gamma_{134,\,256} = 0.$$

These 270 linear relations can be verified by using the above five typical ones in connection with Tables I–II.

Not more than 10 of the 40 complex invariants are linearly independent, for they are expressed above in terms of $d_2\,\bar{a}, \cdots, d_2\bar{f}, \bar{\alpha}, \cdots, \bar{\zeta}$ where $d_2\sum\bar{a} = 0$ and $\sum\bar{\alpha} = 0$. That precisely 10 are independent can be verified by computing $d_2(\bar{a} - \bar{b}), \cdots, d_2(\bar{a} - \bar{f}), \bar{\alpha} - \bar{\beta}, \cdots, \bar{\alpha} - \bar{\zeta}$, and checking their independence.

The group $G_{6,\,2}$ is generated by transpositions of the points and by the element A_{123}. The effect of the transpositions upon the complex invariants can be inferred from the subscript notation in their definitions (16), (17) and (18), (19). The effect of the quadratic transformation A_{123} upon the double sixes and the complexes is easily given. The element A_{123} can at most permute the complex invariants to within outstanding factors. It must however transform the identities 1°, $\cdots$, 5° into identities. If then this outstanding factor is taken to be unity for some one invariant the value of the factor for other invariants can be inferred from the fact that identities involving this one invariant must be transformed into other identities. In

* The table can be completed so readily that it is not given here in full. However a complete table is necessary for checking some of the relations given below.

this way the effect of the involution A_{123} upon the double sixes, the complexes, and the complex invariants is found to be:

$$(D_{123}),\ (D, D_{456}),\ (D_{124}, D_{34}),\ (D_{145}),\ (D_{12}),\ (D_{56});$$

$$(23)\ (\Gamma_{123,\,456}),\ (\Gamma_{124,\,356}, \Gamma_{12,\,34,\,56}),\ (\Gamma_{14,\,25,\,36}, \Gamma_{16,\,35,\,24}),\ (\Gamma_{12,\,56,\,34});$$

$$(\gamma_{123,\,456}),\ (\gamma_{124,\,356}, \gamma_{12,\,34,\,56}),\ (\gamma_{14,\,25,\,36}, -\gamma_{16,\,35,\,24}),\ (\gamma_{12,\,56,\,34}).$$

Of course in these permutations (23) only typical cycles with reference to the division $123, 456$ are given. The permutation of the D's is even while that of the Γ's is odd.

The complex invariants when expressed in terms of the coördinates x, y, z, t, u of P in Σ_4 lie in a linear system of dimension 9 of spreads invariant under $G_{6,\,2}$. Moreover they constitute the simplest linear system of this sort. For a member of such a system invariant under $G_{6!}$ must be expressible in terms of $\bar{a}, \cdots, \bar{f}$. These are quadrics (I, p. 196) on the canonical basis of Σ_4. On the other hand a spread invariant under A_{123} must be of order $5k$ with $3k$-fold points at the reference basis in Σ_4. The simplest way to reconcile these requirements is to take functions of the third degree in $\bar{a}, \cdots, \bar{f}$ from which u factors out. Since d_2 contains the factor u the invariants of Table I have this factor. The factor u appears also in $(156), (256), (125), (126)$ and every complex invariant of the type (18) must contain at least one of these factors. The word "simplest" as used above means of lowest degree in the coördinates of P_6^2.

(24) *The* 40 *complex invariants satisfy a set of* 270 *four termed linear relations by means of which they reduce to* 10 *linearly independent invariants. The latter determine the simplest linear system of irrational invariants of* $C^{(3)}$ *and, as quintic spreads in* Σ_4, *the simplest linear system invariant under* $G_{6,\,2}$. *If* Σ_4 *be mapped upon an* S_9 *by means of this linear system* $G_{6,\,2}$ *appears in* S_9 *as a collineation group* γ_{51840}.

In order to prove that the group γ_{51840} in S_9 has an invariant pair of skew S_4's we shall derive two new sets each containing 45 irrational "tritangent" invariants. A *syzygetic n-ad* of double sixes is a set of n double sixes such that any two are syzygetic. The typical forms of the syzygetic tetrads are

$$D, D_{ij}, D_{kl}, D_{mn};\quad D_{ijk}, D_{ilm}, D_{njl}, D_{nkm};\quad D_{ij}, D_{kl}, D_{ijm}, D_{klm};$$

there being 15, 30, 90 respectively of each kind. The syzygetic duads and triads can all be picked from these tetrads. The 135 syzygetic tetrads divide into 45 sets T of three such that, in any set of three, double sixes from different tetrads are azygetic. Any tetrad lies in but one set T since the 8 double sixes in the other two tetrads of T comprise all that are azygetic with the four

members of the given tetrad. The typical forms of the sets T are

$$T_{ij,\,kl,\,mn} \equiv D\,,\, D_{ij}\,,\, D_{kl}\,,\, D_{mn};\quad D_{ikm}\,,\, D_{iln}\,,\, D_{jkn}\,,\, D_{jlm};$$
$$D_{jkm}\,,\, D_{jln}\,,\, D_{ikn}\,,\, D_{ikm};$$
$$T_{ijkl,\,m} \equiv D_{ij}\,,\, D_{kl}\,,\, D_{ijm}\,,\, D_{klm};\quad D_{ik}\,,\, D_{jl}\,,\, D_{ikm}\,,\, D_{jlm};$$
$$D_{il}\,,\, D_{jk}\,,\, D_{ilm}\,,\, D_{jkm};$$

there being 15 of the first kind and 30 of the second kind. The 48 lines of $C^{(3)}$ in a syzygetic tetrad consist of 24 lines each occurring twice, so that three lines of $C^{(3)}$ *in a tritangent plane* are omitted from a tetrad and the same three lines are omitted from each of the tetrads in T, whence the sets T correspond to the tritangent planes.

A syzygetic triad can be enlarged in but one way into a syzygetic tetrad, whence there are $135 \cdot 4$ syzygetic triads. The four complexes in each of the 270 relations $1°, \cdots, 5°$ each have a common syzygetic duad, whence all the relations are conjugate. We see from $1°$ that any two complexes in such a relation have a common syzygetic triad, whence any two terms must occur in three relations. The two complementary pairs drawn from a relation determine two syzygetic triads drawn from the same tetrad, whence given a tetrad the four triads formed from it determine four pairs of terms and by equating these pairs of terms six relations are obtained. Thus the 135 tetrads determine $135 \cdot 6$ relations, but each is obtained three times, corresponding to its three pairs of pairs of terms.

Let us select the particular set $T_{12,\,34,\,56}$ and write down for each of its tetrads the four equal pairs of terms. We find with the help of Table I and II that

$$
\begin{aligned}
d_2(\bar{d} - \bar{a}) &= \gamma_{12,\,34,\,56} - \gamma_{12,\,56,\,34} = \gamma_{134,\,256} - \gamma_{156,\,234} \\
&= \gamma_{356,\,412} - \gamma_{312,\,456} = \gamma_{512,\,634} - \gamma_{534,\,612}; \\
\tfrac{1}{2}[\bar{\alpha} - \bar{\delta} + d_2(\bar{a} - \bar{d})] &= \gamma_{13,\,46,\,25} - \gamma_{16,\,35,\,24} = \gamma_{14,\,35,\,26} - \gamma_{15,\,46,\,23} \\
&= \gamma_{15,\,24,\,36} - \gamma_{13,\,26,\,45} = \gamma_{16,\,23,\,45} - \gamma_{14,\,25,\,63}; \\
\tfrac{1}{2}[\bar{\delta} - \bar{\alpha} + d_2(\bar{a} - \bar{d})] &= \gamma_{14,\,36,\,25} - \gamma_{16,\,45,\,23} = \gamma_{13,\,45,\,26} - \gamma_{15,\,36,\,24} \\
&= \gamma_{15,\,23,\,46} - \gamma_{14,\,26,\,35} = \gamma_{16,\,24,\,35} - \gamma_{13,\,25,\,64}.
\end{aligned}
\tag{25}
$$

The sum of these three "tetrad invariants" is zero and we form from them as follows the pair of "tritangent invariants":

$$
\begin{aligned}
2(\rho - \rho^2)\,t_{12,\,34,\,56} &= 2[d_2(\bar{d} - \bar{a})] + \rho[\bar{\alpha} - \bar{\delta} + d_2(\bar{a} - \bar{d})] \\
&\qquad + \rho^2[\bar{\delta} - \bar{\alpha} + d_2(\bar{a} - \bar{d})], \\
2(\rho^2 - \rho)\,\tau_{12,\,34,\,56} &= 2[d_2(\bar{d} - \bar{a})] + \rho^2[\bar{\alpha} - \bar{\delta} + d_2(\bar{a} - \bar{d})] \\
&\qquad + \rho[\bar{\delta} - \bar{\alpha} + d_2(\bar{a} - \bar{d})], \\
\rho &= e^{2\pi i/3}.
\end{aligned}
\tag{26}
$$

Both $t_{12,\,34,\,56}$ and $\tau_{12,\,34,\,56}$ contain the factor $\bar{d} - \bar{a}$ since

$$\bar{\alpha} - \bar{\delta} = (\bar{d} - \bar{a})\overline{ad}, \tag{27}$$

and both vanish when the three lines of their corresponding tritangent plane of $C^{(3)}$ meet in a point.

In order to obtain the second type of tritangent invariant we carry out on the above type the transformation A_{123} and interchange t, τ and ρ, ρ^2. Then $T_{12,\,34,\,56}$ becomes $T_{1256,\,4}$ and the three tetrad invariants associated with $T_{1256,\,4}$ become

$$\gamma_{124,\,356} - \gamma_{12,\,56,\,34}, \qquad \gamma_{16,\,23,\,45} + \gamma_{16,\,35,\,24}, \qquad - \gamma_{15,\,34,\,26} - \gamma_{135,\,246}.$$

These have the values

$$\begin{aligned} &- \tfrac{1}{2}[\bar{\alpha} + \bar{\delta} + d_2(\bar{b} + \bar{c} - \bar{e} - \bar{f})], \\ &- \tfrac{1}{2}[\bar{\beta} + \bar{\gamma} + d_2(\bar{e} + \bar{f} - \bar{a} - \bar{d})], \\ &- \tfrac{1}{2}[\bar{\epsilon} + \bar{\zeta} + d_2(\bar{a} + \bar{d} - \bar{b} - \bar{c})]. \end{aligned} \tag{28}$$

As we should expect their sum is zero. With multipliers 1, ρ, ρ^2 respectively we form $(\rho^2 - \rho)\tau_{1256,\,4}$; and with multipliers 1, ρ^2, ρ we form $(\rho - \rho^2)t_{1256,\,4}$. Then

$$\begin{aligned} 2t_{1256,\,4} = [&- \rho(\bar{\beta} + \bar{\gamma}) + \rho^2(\bar{\epsilon} + \bar{\zeta}) - d_2(\bar{a} + \bar{d}) \\ &- \rho^2 d_2(\bar{b} + \bar{c}) - \rho d_2(\bar{e} + \bar{f})], \\ 2\tau_{1256,\,4} = [&- \rho^2(\bar{\beta} + \bar{\gamma}) + \rho(\bar{\epsilon} + \bar{\zeta}) - d_2(\bar{a} + \bar{d}) \\ &- \rho d_2(\bar{b} + \bar{c}) - \rho^2 d_2(\bar{e} + \bar{f})]. \end{aligned} \tag{29}$$

We shall now make an important *change of variables*. Let

$$\begin{aligned} x_a &= \bar{\alpha} + (\rho - \rho^2) d_2 \bar{a}, \quad \cdots, \quad x_f = \bar{\zeta} + (\rho - \rho^2) d_2 \bar{f}, \\ u_a &= \bar{\alpha} + (\rho^2 - \rho) d_2 \bar{a}, \quad \cdots, \quad u_f = \bar{\zeta} + (\rho^2 - \rho) d_2 \bar{f}. \end{aligned} \tag{30}$$

Then the two types of tritangent invariants and the three types of complex invariants take the following very simple forms:

$$\begin{aligned} 2t_{12,\,34,\,56} &= x_a - x_d, \\ 2\tau_{12,\,34,\,56} &= u_a - u_d. \end{aligned} \tag{31}$$

$$\begin{aligned} 2t_{1256,\,4} &= - \rho(x_b + x_c) + \rho^2(x_e + x_f), \\ 2\tau_{1256,\,4} &= - \rho^2(u_b + u_c) + \rho(u_e + u_f). \end{aligned} \tag{32}$$

$$\begin{aligned} 2(\rho - \rho^2)\gamma_{123,\,456} &= x_a + x_b + x_c - u_a - u_b - u_c, \\ 2(\rho - \rho^2)\gamma_{12,\,34,\,56} &= \rho x_a - \rho^2 x_d - \rho^2 u_a + \rho u_d, \\ 2(\rho - \rho^2)\gamma_{12,\,56,\,34} &= - \rho^2 x_a + \rho x_d + \rho u_a - \rho^2 u_d. \end{aligned} \tag{33}$$

The new variables are subject to the relations

$$x_a + \cdots + x_f = 0, \qquad u_a + \cdots + u_f = 0. \tag{34}$$

Their permutations under the operations of $G_{6!}$ are evident at once from (30). In order to determine the effect upon them of the element A_{123} we need only to take the transformation (23), translate it to the new variables by using (33), and express the transformed variables in terms of the original ones. This presents no difficulty and the result is

$$\begin{aligned}
6x'_a &= (-3\rho^2 + \rho)u_a + (3\rho^2 + \rho)(u_b + u_c) + (u_d + u_e + u_f),\\
6x'_b &= (-3\rho^2 + \rho)u_b + (3\rho^2 + \rho)(u_c + u_a) + (u_d + u_e + u_f),\\
6x'_c &= (-3\rho^2 + \rho)u_c + (3\rho^2 + \rho)(u_a + u_b) + (u_d + u_e + u_f),\\
6x'_d &= (u_a + u_b + u_c) + (-3\rho + \rho^2)u_d + (3\rho + \rho^2)(u_e + u_f),\\
6x'_e &= (u_a + u_b + u_c) + (-3\rho + \rho^2)u_e + (3\rho + \rho^2)(u_f + u_d),\\
6x'_f &= (u_a + u_b + u_c) + (-3\rho + \rho^2)u_f + (3\rho + \rho^2)(u_d + u_e).
\end{aligned} \tag{35}$$

The transformation is completed by assigning to the matrix of the transformation which expresses $6u'_a, \cdots, 6u'_f$ in terms of $x_a, \cdots, x_f$ coefficients which are conjugate to those of the matrix in (35).

The most important result which appears from this form of A_{123} is that the variables x and u are interchanged. It is clear from (30) that the same is true of the transpositions of $G_{6!}$. Hence the space S_4 defined in S_9 by

$$u_a = \cdots = u_f = 0,$$

in which $x_a, \cdots, x_f$ are variables and the space $\bar{S}_4$ defined in S_9 by

$$x_a = \cdots = x_f = 0,$$

in which $u_a, \cdots, u_f$ are variables, are either unaltered or interchanged by an operation of γ_{51840} according as this operation can be expressed as a product of an even or an odd number of the 36 generating involutions. Hence

(36) *The collineation group* γ_{51840} *of* (24) *has an invariant pair of skew* S_4'*s. Its invariant subgroup* γ_{25920} *appears in either* S_4 *as a collineation group with a conjugate set of* 40 *linear spaces* (33) *and a conjugate set of* 45 *linear spaces* ((32), (31)).

It is clear from (17), (19), and (23) that

$$\sum_{40} \gamma^2_{123,\,456}\text{*}$$

is an invariant of $G_{6,\,2}$ if it does not vanish identically. From (33) we have

$$-12 \sum_{40} \gamma^2_{123,\,456} = 5 \sum_{6} x_a^2 + 5 \sum_{6} u_a^2 + 4 \sum_{15} x_a x_b + 4 \sum_{15} u_a u_b + 6 \sum_{30} x_a u_b.$$

* The number under a summation sign is the number of terms used in the sum.

If we make use of (34) we find that

$$\sum_{40} \gamma_{123,\,456}^2 = 2 \sum_{6} x_a\, u_a .$$

The expression for this invariant in terms of $\bar{a}, \cdots, \bar{f}$ is from (30):

$$\sum_{6} x_a\, u_a = \sum_{6} (\bar{\alpha}^2 + 3d_2^2\, \bar{a}^2).$$

If we square $\bar{\alpha}$ and use a table of symmetric functions we find in the notation of (9), § 1, that

(37) $$\sum_{6} \bar{\alpha}^2 = 2I_6 + 6a_2\, q_4 .$$

(38) $$\therefore \sum_{40} \gamma_{123,\,456}^2 = 2 \sum_{6} x_a\, u_a = 4I_6 .$$

From this result there follows that under γ_{25920} the variables x and u are *contragredient*. If then x is a point in S_4, u may be regarded as an S_3 in S_4 and the x, u we shall define to be a *counter-point* in S_4. Then the elements of γ_{51840} not in γ_{25920} are correlations in S_4 since they interchange the members of counter-points and have the invariant (38). Thus γ_{51840} can be regarded as a correlation group in S_4, γ_{25920} as the invariant collineation subgroup of this correlation group.

This collineation group in S_4 is reasonably well identified with the Burkhardt group in S_4 by the two conjugate sets described in (36). To do this more precisely we observe with Burkhardt (BII, § 45) that his group can be so represented that it contains a subgroup $G_{6!}$ which permutes symmetrically $6S_3$'s.* If this transformation be applied to his simplest invariant J_4 (BII, p. 208) it takes the form

(39) $$J_4 = \sum_{15} x_a\, x_b\, x_c\, x_d \qquad (x_a + \cdots + x_f = 0).$$

I had noted in using another canonical form† that this quartic spread has a conjugate set of 45 double points. The spread (39) has double points of types

$$1,\ -1,\ 0,\ 0,\ 0,\ 0;\quad 1,\ 1,\ \rho,\ \rho,\ \rho^2\ \rho^2.$$

If we compare these coördinates with the coefficients of the tritangent invariants in (31) and (32), after modifying $2\tau_{1256,\,4}$ by adding

$$-\tfrac{1}{3}(\rho - \rho^2)(u_a + \cdots + u_f)$$

so that the sum of its coefficients is zero, we find them to be the same. Hence the group of the 45 double points is the group of the 45 tritangent invariants τ.

(40) *The Cremona group $G_{6,\,2}$ in Σ_4 effects within its simplest invariant linear system a group of linear transformations γ_{51840} which permutes the irrational*

* Such a subgroup (one of 36) is generated by the elements $(12)\cdot I, \cdots, (56)\cdot I$ of $\Gamma_{6,\,2}$.

† Coble, *An invariant condition for certain automorphic algebraic forms*, American Journal of Mathematics, vol. 28 (1906).

invariants of $C^{(3)}$ *in this linear system as the counter-points in* S_4 *are permuted under the operations of the* correlation group of order* 51840 *built upon Burkhardt's collineation group* G_{25920}.

The group γ_{51840} in S_9 has the invariant (38) which may be regarded as a quadric and which determines in S_9 a polarity P. This polarity and γ_{51840} generate a correlation group $\gamma_{2\cdot 51840}$ in S_9 which has for invariant subgroups both γ_{51840} and the G_2 whose elements are $1, P$. Any element of γ_{51840} is permutable with P whence any involution of γ_{51840} multiplied by P is a polarity Q. A convenient way to indicate any involution of γ_{51840} is therefore to write the equation of the quadric Q. The involution is then QP, a product which can be written at once because of the simple form of P.

In particular the conjugate set of 36 generating involutions of γ_{51840} give rise to the set of Q's:

$$
(41)\quad
\begin{aligned}
\tfrac{1}{6}Q &= x_a^2 + \cdots + x_f^2 + u_a^2 + \cdots + u_f^2,\\
\tfrac{1}{6}Q_{12} &= -2(x_a x_d + x_b x_e + x_c x_f) - 2(u_a u_d + u_b u_e + u_c u_f),\\
\tfrac{1}{6}Q_{456} &= -2\rho^2(x_b x_c + x_c x_a + x_a x_b) - 2\rho(x_e x_f + x_f x_d + x_d x_e)\\
&\quad - 2\rho(u_b u_c + u_c u_a + u_a u_b) - 2\rho^2(u_e u_f + u_f u_d + u_d u_e).
\end{aligned}
$$

It is to be observed that the 36 products PQ, which in S_9 are collineations, in S_4 are correlations, and in fact polarities. The quadrics in S_4 which determine these 36 polarities in S_4 which generate the correlation group (40) are precisely those of (41) where however each Q is the sum of the point-equation and the S_3-equation of the same quadric.

The following theorem, naturally of prime importance, is a consequence of (40), (30), and § 1 (13).

(42) *Any invariant of the Burkhardt group* G_{25920} *of total degree* $2k$ *in* x *and* u, *or the sum or difference of two dual invariants of such total degree, is an invariant of* $C^{(3)}$ *of degree* $6k$ *in* $\bar{a}, \cdots, \bar{f}$, *which is rational in* $I_6, I_{12}, \sqrt{\Delta}, I_{18}$, *and* I_{30}.

In this way invariants of the hexahedral surface can be calculated. For example, we have already found in (38) that the invariant $\sum_6 x_a u_a$ of G_{29520} gives rise to the invariant $2I_6$ of $C^{(3)}$. We find also that the invariant J_4 of (39) and its dual give rise to the following invariants of $C^{(3)}$:

$$
(43)\quad
\begin{aligned}
6\Big[\sum_{15} x_a x_b x_c x_d + \sum_{15} u_a u_b u_c u_d\Big] &= 4I_6^2 + 24a_2 q_4 I_6\\
&\quad + 24a_2 q_5^2 - 288a_3 q_4 q_5 + 80a_2^2 q_4^2 - 48q_4^3,\\
6\Big[\sum_{15} x_a x_b x_c x_d - \sum_{15} u_a u_b u_c u_d\Big] &= -40(\rho - \rho^2)\sqrt{\Delta}.
\end{aligned}
$$

The first of these might be taken as I_{12}, but it is to be remarked that invariants of $C^{(3)}$ obtained in this way are not in immediate relation with a system obtained by known covariant processes such as is contemplated in § 2.

* Since G_{25920} is simple it can be enlarged to a correlation group in only one way.

4. The solution of the form problem of $G_{6,2}$ in terms of a solution of Burkhardt's form problem

The data necessary for this purpose are a set of independent invariants of the Burkhardt group (which we shall take as given in BII with variables $y_0, y_1, \cdots, y_4$) and a set of five invariant mixed forms linear in the space coördinates $v_0, v_1, \cdots, v_4$. The expressions in y for the invariants J_4, J_6, J_{10}, J_{12}, and J_{18} given by Burkhardt seem open to suspicion (cf. footnotes, §§4, 6) and we shall merely assume that independent invariants of the degrees indicated exist which together with their jacobian, J_{45}, constitute a complete system in the variables y alone.

It is noted in BII, p. 218, that the forty squares

$$9y_0^2, \qquad -3(y_0+2\epsilon^\lambda y_\alpha)^2, \qquad (\epsilon=e^{2\pi i/3}),\ (\alpha=1,\cdots,4),$$

$$(y_0+2\epsilon^\lambda y_1+2\epsilon^\mu y_2+2\epsilon^\nu y_3+2\epsilon^{-\lambda-\mu-\nu}y_4)^2 \qquad (\lambda,\mu,\nu=0,1,2),$$

are merely permuted under the operations of the group. If then J_{2k} is the sum of the kth powers of these squares, J_{2k} either vanishes or is an invariant. In order to prove that J_{2k} $(k=2,3,5,6,9)$ together with J_{45} constitute a complete system we have only to prove that their jacobian is not zero and therefore must be a numerical multiple of J_{45}. To simplify the calculation we shall assume after taking the derivatives that

$$y_0=y_4=0, \qquad y_1=-y_2=1, \qquad y_3=t.$$

Then J_{45}, the product of the 45 linear forms given in BII, p. 195, takes the simple form

$$J_{45}=\sigma t^{12}(t^6-1)(t^6+3^3)^3,$$

where σ is a numerical constant. The first derivatives of J_{2k} for the given values of k become

$$\frac{1}{2^{2k-1}}\frac{1}{2k}\frac{\partial J_{2k}}{\partial y_0}=\sum_\lambda(-3)^k\epsilon^{\lambda(2k-1)}t^{2k-1}+\sum_{\lambda,\mu,\nu}(\epsilon^\lambda-\epsilon^\mu+\epsilon^\nu t)^{2k-1},$$

$$\frac{1}{2^{2k}}\frac{1}{2k}\frac{\partial J_{2k}}{\partial y_1}=\sum_\lambda(-3)^k\epsilon^{2k\lambda}+\sum_{\lambda,\mu,\nu}(\epsilon^\lambda-\epsilon^\mu+\epsilon^\nu t)^{2k-1}\epsilon^\lambda,$$

$$\frac{1}{2^{2k}}\frac{1}{2k}\frac{\partial J_{2k}}{\partial y_2}=\sum_\lambda-(-3)^k\epsilon^{2k\lambda}+\sum_{\lambda,\mu,\nu}(\epsilon^\lambda-\epsilon^\mu+\epsilon^\nu t)^{2k-1}\epsilon^\mu,$$

$$\frac{1}{2^{2k}}\frac{1}{2k}\frac{\partial J_{2k}}{\partial y_3}=\sum_\lambda(-3)^k\epsilon^{2k\lambda}t^{2k-1}+\sum_{\lambda,\mu,\nu}(\epsilon^\lambda-\epsilon^\mu+\epsilon^\nu t)^{2k-1}\epsilon^\nu,$$

$$\frac{1}{2^{2k}}\frac{1}{2k}\frac{\partial J_{2k}}{\partial y_4}=\sum_{\lambda,\mu,\nu}(\epsilon^\lambda-\epsilon^\mu+\epsilon^\nu t)^{2k-1}\epsilon^{-\lambda-\mu-\nu}.$$

We see that for $k=2$ and $k=5$ only the first and last derivatives do not

vanish so that the determinant of these derivatives is a factor of the jacobian which turns out to be

$$-2^3\cdot 3^7\cdot 5^2\, t^4(t^6+3^3),$$

which is a factor of J_{45}. The remaining factor is the determinant of the other derivatives for $k = 3, 6, 9$. If we write these derivatives in order for a given k in a column, then the first and second rows are interchanged with a change of sign throughout, if λ and μ be interchanged and t changed in sign. If then we take the sum and difference of the first two rows for new first two rows, t^3 factors from the new first row and t^5 factors from the last row. The factor t^{12} of J_{45} is now accounted for and the columns are of degrees $0, 1, 2$ in t^6. It will now be sufficient to prove that the coefficient of the highest power of t^6 is not zero. This is

$$27^3\cdot 2^2\begin{vmatrix} 10 & \binom{11}{2} & \binom{17}{2} \\ -12 & -9\binom{11}{5} & -9\binom{17}{5} \\ 1-3 & 1+3^4 & 1-3^7 \end{vmatrix} = 2^5\cdot 3^{12}\cdot 5\cdot 31\cdot 4181.$$

(44) *The invariants $J_{2\cdot 2}$, $J_{2\cdot 3}$, $J_{2\cdot 5}$, $J_{2\cdot 6}$, and $J_{2\cdot 9}$ defined above, together with J_{45}, constitute a complete system for the Burkhardt group.*

We shall indicate by $J_{i,\,k}$ an invariant form of this group of degree i in y and of degree k in the dual coördinates v. Then the group has an invariant $J_{0,\,4}$, the dual of $J_{4,\,0}$ found above. I had noted (see loc. cit., § 3) that these two forms had the symmetrical property that the polar quadric of a given quadric as to either would have for polar as to the other the given quadric. Thus either form defines the other and if we take with Burkhardt

$$J_{4,\,0} = y_0^4 + 8y_0(y_1^3+\cdots+y_4^3) + 48y_1y_2y_3y_4,$$

then $J_{0,\,4}$ must be

$$J_{0,\,4} = v_0^4 + v_0(v_1^3+\cdots+v_4^3) + 3v_1v_2v_3v_4.$$

If we operate with the polar cubic of v as to $J_{0,\,4}$ on any invariant $J_{r,\,0}$ the result, if not zero, is an invariant $J_{r-3,\,1}$. We shall now prove that

(45) *If we operate with the polar cubic of v as to $J_{0,\,4}$ upon the sums of powers J_{2k} for $k = 2, 5, 6, 8, 9$ we obtain five forms $J_{1,\,1}$, $J_{7,\,1}$, $J_{9,\,1}$, $J_{13,\,1}$, and $J_{15,\,1}$ which in v are linearly independent and have for determinant J_{45}.*

The proof will be carried out as above. The coefficients of $v_0, \cdots, v_4$ in $J_{2k-3,\,1}$ become after using the above special values of y and taking out the numerical factor $2k(2k-1)(2k-2)2^{2k-1}\cdot 3$

$$\sum_{\lambda}(-3)^k\epsilon^{2k\lambda}t^{2k-3} \quad + 3\sum_{\lambda,\,\mu,\,\nu}(\epsilon^\lambda-\epsilon^\mu+\epsilon^\nu t)^{2k-3},$$

$$\sum_{\lambda}(-3)^k\epsilon^{(2k-1)\lambda} \quad + 3\sum_{\lambda,\,\mu,\,\nu}(\epsilon^\lambda-\epsilon^\mu+\epsilon^\nu t)^{2k-3}\epsilon^{2\lambda},$$

$$\sum_{\lambda} -(-3)^k \epsilon^{(2k-1)\lambda} + 3\sum_{\lambda,\mu,\nu}(\epsilon^{\lambda} - \epsilon^{\mu} + \epsilon^{\nu} t)^{2k-3}\epsilon^{2\mu},$$

$$\sum_{\lambda} (-3)^k \epsilon^{(2k-1)\lambda} t^{2k-3} + 3\sum_{\lambda,\mu,\nu}(\epsilon^{\lambda} - \epsilon^{\mu} + \epsilon^{\nu} t)^{2k-3}\epsilon^{2\nu}$$
$$+ 3\sum_{\lambda,\mu,\nu}(\epsilon^{\lambda} - \epsilon^{\mu} + \epsilon^{\nu} t)^{2k-3}\epsilon^{\lambda+\mu+\nu}.$$

Again for $k = 6, 9$ the coefficients of v_1, v_2, v_3 vanish so that the determinant of the five forms contains as a factor the two-rowed determinant of coefficients of v_0, v_4 for $k = 6, 9$. This is easily reduced to

$$3^6\begin{vmatrix} (3^4+3)t^9 - 2\cdot 3^3\binom{9}{3}t^3 & 6\binom{9}{4}t^5(\epsilon-\epsilon^2)^4 \\ (-3^7+3)t^{15} - 2\cdot 3^3\binom{15}{9}t^9 + 2\cdot 3^6\binom{15}{3}t^3 & 6\binom{15}{4}t^{11}(\epsilon-\epsilon^2)^4 + 6\binom{15}{10}t^5(\epsilon-\epsilon^2)^{10} \end{vmatrix}$$

$$= 3^{11}\cdot 2^2\cdot 7^2\cdot 13t^8\begin{vmatrix} t^6 - 2\cdot 3^3 & 3 \\ -4t^{12} - 3^2\cdot 5\cdot 11t^6 + 3^5\cdot 5 & 5t^6 - 3^3\cdot 11 \end{vmatrix}$$
$$= 3^{11}\cdot 2^2\cdot 7^2\cdot 13\cdot 17t^8(t^6+3^3)^2.$$

This is a factor of J_{45}.* There remains the determinant of the coefficients of v_1, v_2, v_3 for $k = 2, 5, 8$ which reads as follows when we denote

$$\sum_{\lambda,\mu}(\epsilon^{\lambda} - \epsilon^{\mu} + t)^k \quad \text{by} \quad s_k:$$

$$\begin{vmatrix} 3^3 + 9s_1\epsilon^{2\lambda} & -3^6 + 9s_7\epsilon^{2\lambda} & 3^9 + 9s_{13}\epsilon^{2\lambda} \\ -3^3 + 9s_1\epsilon^{2\mu} & 3^6 + 9s_7\epsilon^{2\mu} & -3^9 + 9s_{13}\epsilon^{2\mu} \\ 3^3t + 9s_1 & -3^6t^7 + 9s_7 & 3^9t^{13} + 9s_{13} \end{vmatrix}.$$

By adding and subtracting the first and second rows to form new rows we get the determinant

$$2^4\cdot 3^{11}t^4\begin{vmatrix} 0 & -35 & -\binom{13}{4}t^6 + \binom{13}{10}3^3 \\ 1 & 3(7t^6 - 36) & 3[\binom{13}{1}t^{12} - \binom{13}{7}t^6 + 4\cdot 3^5] \\ 1 & -3(8t^6 + 18\cdot 7) & 3\cdot 244t^{12} - 2\binom{13}{6}t^6 + 2\cdot 13\cdot 3^6 \end{vmatrix}$$
$$= 2^5\cdot 3^{14}\cdot 5\cdot 11\cdot 19t^4(t^6-1)(t^6+3^3)$$

which is the remaining factor of J_{45} and theorem (45) is established.

According to § 3 (42) the invariants $J_{2\cdot 2}, \cdots, J_{2\cdot 9}$ of (44) give rise to in-

* I originally tried to derive a $J_{9,1}$ and $J_{15,1}$ by operating with the above polar cubic on Burkhardt's J_{12} and J_{18} as explicitly given in terms of the y's in BII, pp. 208–9. As above only the terms in v_0, v_4 persisted for $y_0 = y_4 = 0$. But their two-rowed determinant could not be expressed as a factor of J_{45}. Either my own calculation was wrong or there is some mistake in one of these forms other than the misprint (p. 209) in the term $-8\Sigma y_1^6 y_2^3 y_3^3 y_4^3$ which should read $-8\Sigma y_1^9 y_2^3 y_3^3 y_4^3$. This was one of the reasons for introducing a new complete system.

variants I'_{12}, I'_{18}, I'_{30}, I'_{36}, and I'_{54} of the group $\Gamma_{6,\,2}$ and the invariants $J_{1,\,1}$, $\cdots$, $J_{15,\,1}$ of (45) give rise to invariants I''_6, I''_{24}, I''_{30}, I''_{42}, and I''_{48} of $\Gamma_{6,\,2}$. Explicit expressions for the invariants I' and I'' could be obtained if a complete system for $G_{6,\,2}$ had been selected. Their expressions in terms of $\bar{a}$, $\cdots$, $\bar{f}$ could easily be written, since they are given in terms of the sums of powers of the 40 linear forms in x found in § 3 (33) and from § 3 (30), (21) the values of x in terms of $\bar{a}$, $\cdots$, $\bar{f}$ are known.

The requisite steps in the solution of the form problem of $\Gamma_{6,\,2}$ when a solution of Burkhardt's form problem is adjoined are as follows:

1°. From the given values of I_6, I_{12}, $\sqrt{\Delta}$, I_{18}, I_{30} the values of I'_{12}, I'_{18}, I'_{30}, I'_{36}, I'_{54} above are determined and thereby the given values of the Burkhardt form problem are ascertained.

2°. From these values a solution of the Burkhardt form problem is calculated.

This solution we suppose taken in the canonical form x_a, $\cdots$, x_f where the factor of proportionality λ in the x's is determined from $J_{2\cdot 3}/J_{2\cdot 2}$ to within sign.

3°. The values of I''_6, $\cdots$, I''_{48} are determined in terms of the invariants of $\Gamma_{6,\,2}$; and from the system of five linear equations

$$J_{1,\,1} = I''_6, \qquad \cdots, \qquad J_{15,\,1} = I''_{48}$$

together with $u_a + \cdots + u_f = 0$, the values of u_a, $\cdots$, u_f are found in terms of the I'' and of x_a, $\cdots$, x_f to within the same factor λ.

4°. From the equations

$$x_a - u_a = 2(\rho - \rho^2)\, d_2\, \bar{a}, \qquad \cdots, \qquad x_f - u_f = 2(\rho - \rho^2)\, d_2\, \bar{f},$$

values of $\mu\bar{a}$, $\cdots$, $\mu\bar{f}$ are obtained. If these values are used in place of $\bar{a}$, $\cdots$, $\bar{f}$ to determine $\overline{\alpha}$ in $x_a + u_a = 2\overline{\alpha}$, the value of μ^3 is rationally determined. If in (8) § 1 we use $\mu\bar{a}$, etc., rather than $\bar{a}$, etc., in forming Π we have

$$\sqrt{\Delta} = \frac{d_2\, \Pi_\mu}{\mu^{10}} = \frac{d_2\, \mu^2\, \Pi_\mu}{\mu^{12}}$$

whence $d_2\,\mu^2$ is rationally determined.

5°. The ratios of the coördinates of the point P in Σ_4 which is a solution of $\Gamma_{6,\,2}$ are obtained from I, § 10 (89) by replacing $\bar{a}$, $\cdots$, $\bar{f}$, ρ by $\mu\bar{a}$, $\cdots$, $\mu\bar{f}$, $\mu\rho$ where $\mu\rho$ is rational in $\mu\bar{a}$, $\cdots$, $\mu\bar{f}$, $\mu^2 d_2$.

The above procedure is valid except in the two cases when either Δ vanishes, or J_{45}, the determinant of the system of linear equations in 3°, vanishes. If Δ vanishes the lines of $C^{(3)}$ can be determined by the solution of a general sextic equation and the above apparatus is unnecessary. If J_{45} vanishes then one of the tritangent invariants t of § 3 (26), (29) must vanish. The product of these is an invariant $I_{75}\cdot I_{60}$ of $\Gamma_{6,\,2}$. Without stopping to discuss

the geometric meaning of I_{60} it is clear that the vanishing of J_{45} must imply the isolation of a tritangent plane of $C^{(3)}$ in which case as noted before the equations of the lines can be expressed by means of radicals.

That the Burkhardt form problem has precisely 25920 solutions shows that the form problem of $\Gamma_{6,\,2}$ has 25920 solutions and therefore that of $G_{6,\,2}$ has 51840 solutions. For any solution for $\Gamma_{6,\,2}$ leads to a solution for G_{25920} and conversely. We now have to consider further only those processes involved in 2°.

5. The normal hyperelliptic surface in S_8 of genus two and grade three

In this paragraph we shall develop certain geometrical facts concerning the hyperelliptic surface and its projections which suggest a solution of the Burkhardt form problem for the special case $J_4 = 0$.

It is known* that there are 3^{2p} essentially distinct theta functions of the first order whose characteristic can be formed from rational numbers with denominator 3. We shall use the $3^{2p} = 81$ $(p = 2)$ cubes of these functions, i. e., 81 theta functions of the third order and characteristic zero. Of these only $3^p = 9$ are linearly independent, so that if such a set of 9 are equated to homogeneous coördinates in a linear space S_8 they furnish, as the variables (u) change a parametric representation of a manifold M_2 in S_8. The order of M_2 is the number of common zeros of two such functions, i. e., $3^2 \cdot 2! = 18$.

If we replace (u) by $(u) + (P)$ where (P) is any period then (K, p. 371, V, VI) the 81 theta cubes all are affected by the same factor of proportionality so that values $(u)' \equiv (u)$ furnish the same point on M_2^{18}. If we replace (u) by $(u + \frac{1}{3}P)$ then (K, p. 371, VII, VIII) the 81 theta cubes are permuted cyclically in sets of three to within a factor of proportionality common to all. Hence $(u)' \equiv (u + \frac{1}{3}P)$ is the parametric expression of a collineation of period three which transforms M_2^{18} into itself. By allowing $(\frac{1}{3}P)$ to take all possible values a collineation G_{81} is obtained. If we replace (u) by $-(u)$ then (K, p. 372, IX) all the theta cubes, except $[\vartheta(u)]^3$ which is unaltered, are interchanged in pairs without extraneous factors. Hence $(u)' \equiv -(u)$ is the parametric expression of an involution I which also transforms M_2^{18} into itself. These permutations of the functions are independent of variation of the moduli $\tau_{11}, \tau_{12}, \tau_{22}$ of the functions, whence

(46) *Any one of a family of* ∞^3 *spreads* M_2^{18} *defines the collineation group* $G_{2\cdot 81}$ *under which each member of the family is invariant. The parametric form of* $G_{2\cdot 81}$ *is* $(u)' = \pm(u) + (\frac{1}{3}P)$. *The* $G_{2\cdot 81}$ *contains an abelian subgroup* G_{81} *of signature* $(3, 3, 3, 3)$ *and also contains* 81 *involutions conjugate to* I *under* G_{81}.

* Cf. Krazer, *Lehrbuch der Thetafunktionen*, pp. 370–72; cited hereafter as K.

We shall speak of M_2^{18} as the *normal hyperelliptic surface of genus* 2 *and grade* 3 since any surface whose parametric equations contain only functions of the third order and zero characteristic is either M_2^{18} or one of its projections.

To obtain a convenient form for $G_{2\cdot 81}$ let us suppose that a member of the family has been isolated for which τ_{12} vanishes. The theta functions of the third order and zero characteristic can then be linearly expressed in terms of products of similar elliptic theta functions with respectively variables u, v and moduli τ_{11}, τ_{22}. If the third periods are $\frac{1}{3}Q$ and $\frac{1}{3}R$ respectively the $G_{2\cdot 81}$ is a product of the parametric substitutions

$$u' \equiv eu + \tfrac{1}{3}Q, \qquad v' \equiv ev + \tfrac{1}{3}R \,(e = \pm 1).$$

As we know such linear combinations of the elliptic thetas can be formed that their collineation group $G_{2\cdot 9}$ is generated by

$$x_i' = x_{i+1}, \qquad x_i' = \rho^i x_i, \qquad x_i' = x_{3-i} \qquad (i \equiv 0, 1, 2 \bmod 3).$$

Similar combinations $\overline{x}_i$ in the variables v can be formed and the 9 products $x_i \overline{x}_j$ define special members of the family (46). The $G_{2\cdot 81}$ of such a special member, and therefore also of the general member, of the family has the generators:

$$(47) \qquad x_{ij}' = x_{i+1,\,j}, \qquad x_{ij}' = \rho^i x_{ij}, \qquad x_{ij}' = x_{i,\,j+1},$$
$$x_{ij}' = \rho^j x_{ij}, \qquad x_{i,\,j}' = x_{3-i,\,3-j}$$
$$(\rho = e^{2\pi i/3};\ i, j \equiv 0, 1, 2 \bmod 3).$$

The set of 81 points $(u) \equiv (\frac{1}{3}P)$ on M_2^{18} is a conjugate set under G_{81}. The particular involution I given by $(u)' \equiv -(u)$ or $x_{ij}' = x_{3-i,\,3-j}$ has 16 fixed points on M_2^{18} namely the one point $(u) \equiv 0$ included in the above set and the 15 points $(u) \equiv (\frac{1}{2}P) \not\equiv 0$. All of these points must be found on the fixed spaces of I. If in order to bring into evidence these fixed spaces we make the change of coördinates

$$(48) \qquad \begin{aligned} y_0 &= x_{00}, & & \\ 2y_1 &= x_{01} + x_{02}, & 2z_1 &= x_{01} - x_{02}, \\ 2y_2 &= x_{10} + x_{20}, & 2z_2 &= x_{10} - x_{20}, \\ 2y_3 &= x_{11} + x_{22}, & 2z_3 &= x_{11} - x_{22}, \\ 2y_4 &= x_{12} + x_{21}, & 2z_4 &= x_{12} - x_{21}, \end{aligned}$$

we find that the spaces of fixed points of I are an S_3 with coördinates z determined by $y = 0$, and an S_4 with coördinates y determined by $z = 0$. Since in the degenerate elliptic case when $u \equiv 0$, $x_0 = x_1 + x_2 = 0$, and when $u \equiv \frac{1}{2}Q$, $x_1 - x_2 = 0$ we see that in general the fixed S_4 contains the point $(u) \equiv 0$ and the 9 points $(u) \equiv (\frac{1}{2}P)$ where $(\frac{1}{2}P)$ has an even characteristic while the fixed S_3 contains the 6 points $(u) \equiv (\frac{1}{2}P)$ where $(\frac{1}{2}P)$ has an odd characteristic. Since a pair of corresponding points of I is projected from either fixed space into a single point of the other fixed space we have

(49) *Each M_2^{18} of the family invariant under the $G_{2\cdot 81}$ generated by* (47) *has a point* $(u) \equiv 0$ *in the fixed S_4 of the involution I of $G_{2\cdot 81}$. This fixed S_4 meets M_2^{18} further in the 9 even half period points and the fixed S_3 meets M_2^{18} in the 6 odd half-period points. The family of ∞^3 M_2^{18}'s is projected from the S_3 into a family of doubly covered spreads N_2^6 in S_4 and is projected from the S_4 into a family of doubly covered surfaces W_2^4 in S_3.*

The general quadratic form in S_8 contains 9 squares and 36 product terms. These all are theta functions of order 6 and characteristic 0 so that there must be at least 9 relations among them which we find as follows. Assume any relation among the 45 terms. Being an identity it must be transformed into an identity by the collineation $x'_{ij} = \rho^j x_{ij}$ and we obtain thus three identities. If these be multiplied respectively by $1, 1, 1$, then by $1, \rho, \rho^2$, then by $1, \rho^2, \rho$ and in each case added there will be obtained three identities each consisting of those terms of the original identity for which the sum of the second subscripts is a constant mod 3. Proceeding in the same way with $x'_{ij} = \rho^i x_{ij}$ we get from each of the three identities three new identities such that in each term the sum of the first subscripts is a constant mod 3. Hence there must be an identity of one of the 9 following forms and beginning with any one the remaining ones are obtained from it by the collineations $x'_{i,j} = x_{i+1,j}$, $x'_{i,j} = x_{i,j+1}$.

$$
(50)\quad
\begin{aligned}
&\alpha_0 x_{00}^2 + 2\alpha_1 x_{01} x_{02} + 2\alpha_2 x_{10} x_{20} + 2\alpha_3 x_{11} x_{22} + 2\alpha_4 x_{12} x_{21} = 0,\\
&\alpha_0 x_{01}^2 + 2\alpha_1 x_{02} x_{00} + 2\alpha_2 x_{11} x_{21} + 2\alpha_3 x_{12} x_{20} + 2\alpha_4 x_{10} x_{22} = 0,\\
&\alpha_0 x_{02}^2 + 2\alpha_1 x_{00} x_{01} + 2\alpha_2 x_{12} x_{22} + 2\alpha_3 x_{10} x_{21} + 2\alpha_4 x_{11} x_{20} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

These are the 9 expected relations with coefficients $\alpha_0, \cdots, \alpha_4$ which are modular forms still to be determined.

If we apply to these quadrics the involution I the first is unaltered while the others are permuted in pairs. If we add and subtract the members of a pair to form a new pair and then introduce the coördinates y and z from (48) we get the new equations

$$
\begin{aligned}
&\alpha_0 y_0^2 + 2\alpha_1 y_1^2 + 2\alpha_2 y_2^2 + 2\alpha_3 y_3^2 + 2\alpha_4 y_4^2\\
&\qquad\qquad - 2\alpha_1 z_1^2 - 2\alpha_2 z_2^2 - 2\alpha_3 z_3^2 - 2\alpha_4 z_4^2 = 0,\\
&\alpha_0 y_1^2 + 2\alpha_1 y_0 y_1 + 2\alpha_2 y_3 y_4 + 2\alpha_3 y_2 y_4 + 2\alpha_4 y_2 y_3\\
&\qquad\qquad + \alpha_0 z_1^2 - 2\alpha_2 z_3 z_4 - 2\alpha_3 z_2 z_4 - 2\alpha_4 z_2 z_3 = 0,\\
&\alpha_0 y_2^2 + 2\alpha_1 y_3 y_4 + 2\alpha_2 y_0 y_2 + 2\alpha_3 y_1 y_4 + 2\alpha_4 y_3 y_1\\
&\qquad\qquad + \alpha_0 z_2^2 + 2\alpha_1 z_3 z_4 \qquad + 2\alpha_3 z_1 z_4 - 2\alpha_4 z_3 z_1 = 0,
\end{aligned}
$$

$$\begin{aligned}(51)\quad &\alpha_0 y_3^2 + 2\alpha_1 y_4 y_2 + 2\alpha_2 y_4 y_1 + 2\alpha_3 y_0 y_3 + 2\alpha_4 y_2 y_1 \\ &\qquad + \alpha_0 z_3^2 + 2\alpha_1 z_4 z_2 - 2\alpha_2 z_4 z_1 + 2\alpha_4 z_2 z_1 = 0, \\ &\alpha_0 y_4^2 + 2\alpha_1 y_2 y_3 + 2\alpha_2 y_3 y_1 + 2\alpha_3 y_2 y_1 + 2\alpha_4 y_0 y_4 \\ &\qquad + \alpha_0 z_4^2 + 2\alpha_1 z_2 z_3 + 2\alpha_2 z_3 z_1 - 2\alpha_3 z_2 z_1 = 0, \\ &z_1 \pi_{01} + z_2 \pi_{43} + z_3 \pi_{24} + z_4 \pi_{32} = 0, \\ &z_1 \pi_{43} + z_2 \pi_{02} + z_3 \pi_{14} + z_4 \pi_{13} = 0, \qquad (\pi_{ik} = \alpha_i y_k - \alpha_k y_i). \\ &z_1 \pi_{24} + z_2 \pi_{14} + z_3 \pi_{03} + z_4 \pi_{12} = 0, \\ &z_1 \pi_{32} + z_2 \pi_{13} + z_3 \pi_{12} + z_4 \pi_{04} = 0,\end{aligned}$$

(52) *The normal surface M_2^{18} is the complete intersection of the nine quadric spreads of* (50) *or* (51).

That it is the complete intersection will follow later from the fact that the projections are completely defined by the above equations.

Since for $(u) \equiv 0$ we have a point on the fixed S_4 of I the coördinates z of this point are zero and from the simplified form of the first five quadrics (51) we see that *the point $(u) \equiv 0$ of M_2^{18} is on the hessian J_{10} and the modular forms $\alpha_0, \cdots, \alpha_4$ are the coördinates of the corresponding point of the steinerian of the quartic spread $J_4 = y_0^4 + 8y_0(y_1^3 + \cdots + y_4^3) + 48y_1 y_2 y_3 y_4$ in the fixed S_4 of I.* However we shall see later that this is only a partial statement of the hessian and steinerian relation. For a general point y, z on M_2^{18} the last four equations (51) hold and the z's are not zero, so that the y's which are also the coördinates in S_4 of a projected point pair of M_2^{18} must satisfy the equation

$$(53)\qquad K = \begin{vmatrix} \pi_{01} & \pi_{43} & \pi_{24} & \pi_{32} \\ \pi_{43} & \pi_{02} & \pi_{14} & \pi_{13} \\ \pi_{24} & \pi_{14} & \pi_{03} & \pi_{12} \\ \pi_{32} & \pi_{13} & \pi_{12} & \pi_{04} \end{vmatrix} = 0.$$

Hence the doubly covered surface N_2^6 lies on this quartic cone K. In order to determine the point α and thereby to locate this cone we observe that for the 6 points of M_2^{18} on S_3 the coördinates y are zero and therefore these points will lie on the five quadrics in S_3 obtained by setting $y = 0$ in (51). If these be multiplied respectively by $\alpha_0, 2\alpha_1, \cdots, 2\alpha_4$ and added the result is zero so that the quadrics

$$\begin{aligned}(54)\quad &\alpha_0 z_1^2 - 2\alpha_2 z_3 z_4 - 2\alpha_3 z_2 z_4 - 2\alpha_4 z_3 z_2 = 0, \\ &\alpha_0 z_2^2 + 2\alpha_1 z_3 z_4 + 2\alpha_3 z_4 z_1 - 2\alpha_4 z_1 z_3 = 0, \\ &\alpha_0 z_3^2 + 2\alpha_1 z_4 z_2 - 2\alpha_2 z_1 z_4 + 2\alpha_4 z_2 z_1 = 0, \\ &\alpha_0 z_4^2 + 2\alpha_1 z_2 z_3 + 2\alpha_2 z_3 z_1 - 2\alpha_3 z_1 z_2 = 0,\end{aligned}$$

meet in the six points, say the P_6^3, in which M_2^{18} cuts S_3. For any one of these points z, equations (54) determine the ratios of the α's to be

$$
(55)\qquad
\begin{aligned}
\alpha_0 &= 6z_1 z_2 z_3 z_4, \\
\alpha_1 &= -z_1(\quad z_2^3 + z_3^3 + z_4^3), \\
\alpha_2 &= z_2(z_1^3 \quad + z_3^3 - z_4^3), \\
\alpha_3 &= z_3(z_1^3 - z_2^3 \quad + z_4^3), \\
\alpha_4 &= z_4(z_1^3 + z_2^3 - z_3^3 \quad).^*
\end{aligned}
$$

Now as the moduli vary the P_6^3 of M_2^{18} on S_3 runs over the S_3 and equations (55) constitute the map of P_6^3's in S_3 upon a certain spread in the fixed S_4. The mapping is effected by a linear system of ∞^4 quartic spreads on a Witting configuration, say a W_{40}, made up of the 40 points

$$
(56)\qquad
\begin{aligned}
z_i &= 1, & z_j &= 0 & (i \neq j;\ i, j = 1, \cdots, 4); \\
z_1 &= 0, & z_2^3 &= z_3^3 = z_4^3; \\
z_2 &= 0, & z_3^3 &= -z_4^3 = z_1^3; \\
z_3 &= 0, & z_4^3 &= z_1^3 = -z_2^3; \\
z_4 &= 0, & z_1^3 &= z_2^3 = -z_3^3.
\end{aligned}
$$

Three of these quartic surfaces meet in 64 points, of which only 24 are variable, but these divide into $4P_6^3$'s so that a line in S_4 meets the map of S_3 under (55) in four points. Since this map must be invariant under the modular group it must be the spread J_4 and it is easy to verify directly that $J_4(\alpha)$ vanishes for the values α given in (55).†

Another interesting fact concerning the system of quartic surfaces on W_{40} arises by considering the jacobian of the four quadrics (54), which is a Weddle

* Cf. BIII, p. 337; the formulas there given are wrong, since as they stand $J_4(Y) \neq 0$. The Y_0 and Y_1 should be changed in sign.

† The modular groups in y and z are fully discussed by Klein, Witting, and Burkhardt. It is clear that the $G_{2\cdot 81}$ contains 40 G_3's and that further collineations exist which permute these G_3's. Thus $G_{2\cdot 81}$ is an invariant subgroup of a collineation $G_{2\cdot 81\cdot m}$. The elements of this group must leave the family of M_2^{18}'s invariant but must permute its members and therefore must arise by adjoining transformations on τ_{ik}. These additional collineations could be determined readily by finding the collineations on the z's which permute the points of W_{40}, by finding from (55) the corresponding collineations on the α's which are cogredient with the y's, and by adjusting the factors of proportionality so that the system (51) in y, z, α is invariant. The modular groups in y alone or z alone are obtained from $G_{2\cdot 81\cdot m}$ by considering the subgroup which leaves I unaltered, a subgroup which is isomorphic with the factor group of $G_{2\cdot 81}$ under $G_{2\cdot 81\cdot m}$.

quartic with nodes at the P_6^3 common to the four. This jacobian is

$$
J \equiv \frac{6}{\alpha_0} \begin{vmatrix} \alpha_0 z_1 & -\alpha_3 z_4 - \alpha_4 z_3 & -\alpha_4 z_2 - \alpha_2 z_4 & -\alpha_2 z_3 - \alpha_3 z_2 \\ \alpha_3 z_4 - \alpha_4 z_3 & \alpha_0 z_2 & \alpha_1 z_4 - \alpha_4 z_1 & \alpha_1 z_3 + \alpha_3 z_1 \\ -\alpha_2 z_4 + \alpha_4 z_2 & \alpha_1 z_4 + \alpha_4 z_1 & \alpha_0 z_3 & \alpha_1 z_2 - \alpha_2 z_1 \\ \alpha_2 z_3 - \alpha_3 z_2 & \alpha_1 z_3 - \alpha_3 z_1 & \alpha_1 z_2 + \alpha_2 z_1 & \alpha_0 z_4 \end{vmatrix}, \tag{57}
$$

$$
\begin{aligned} J = {} & [\alpha_0^3 + 2(\alpha_1^3 + \cdots + \alpha_4^3)]\, 6 z_1 z_2 z_3 z_4 \\ & + 6(\alpha_0 \alpha_1^2 + 2\alpha_2 \alpha_3 \alpha_4)[-z_1(z_2^3 + z_3^3 + z_4^3)] \\ & + \cdots + 6(\alpha_0 \alpha_4^2 + 2\alpha_1 \alpha_2 \alpha_3)[z_4(z_1^3 + z_2^3 - z_3^3). \end{aligned}
$$

If then we write J_4 in symbolic form

$$J_4 = (ay)^4 = (by)^4 = \cdots, \tag{58}$$

we have the simple equation for J

$$J = (a\alpha)^3 (a\alpha'), \tag{59}$$

where α' can be expressed as a quartic in z by means of (55).

(60) *The fixed S_3 of I is mapped in* (55) *upon the fixed S_4 by quartic spreads on W_{40}. The sets P_6^3 cut out on S_3 by the ∞^3 M_2^{18}'s map upon single points of the quartic spread J_4 in S_4 which has the peculiarity of being its own steinerian. If P_6^3 maps upon α on J_4 the tangent S_3 to J_4 at α cuts J_4 in the map of a Weddle quartic in S_3 with nodes at P_6^3 and in the linear system on W_{40}. Thus the linear system contains ∞^3 Weddle surfaces one with a node at each point of S_3.*

The following useful identity can be verified easily:

$$(a\alpha)^4 \cdot (ay)^4 - 4(a\alpha)^3(ay) \cdot (a\alpha)(ay)^3 + 3[(a\alpha)^2(ay)^2]^2 = 48K; \tag{61}$$

i. e.,

(62) *The locus of lines in S_4 which cut the quartic spread J_4 in four self-apolar points is the quartic complex K.*

We have already noted that the doubly covered N_2^6 lies on the cone K for given α on J_4. If we multiply the first five quadrics (51) by $\alpha_0, 2\alpha_1, \cdots, 2\alpha_4$ respectively and add, the z's disappear and we have $(a\alpha)^2(ay)^2 = 0$. Hence N_2^6 lies on the polar quadric of α as to J_4. This meets K in an octavic 2-way. But from (61) if $(a\alpha)^4 = 0$, $(a\alpha)^2(ay)^2 = 0$, and $K = 0$, then either $(a\alpha)^3(ay) = 0$ or $(a\alpha)(ay)^3 = 0$. The octavic 2-way breaks up into a quadric and N_2^6, which is the complete intersection of $(a\alpha)^2(ay)^2 = 0$ and $(a\alpha)(ay)^3 = 0$. If π is any plane on α, π meets J_4 in a quartic curve on α and meets $(a\alpha)(ay)^3 = 0$ and $(a\alpha)^2(ay)^2 = 0$ in the polar cubic and conic of α as to this curve. The polar curves meet in only four points outside α

and from (61) the lines from α to these four points are on K, whence N_2^6 has a double point α and its cone of projection from α is K.

The argument used above to show that $(u) \equiv 0$ furnishes a point of the hessian J_{10} whose steinerian point α is on J_4 applies equally well to show that any one of the 10 points of M_2^{18} on S_4 lies on J_{10} and that all have the same steinerian point α. Hence the polar cubic of α on J_4 as to J_4 has 10 nodes on J_{10}. But a cubic spread in S_4 with 10 nodes is a Segre cubic* and its enveloping cone from a point on it is a Kummer cone, i. e., a cone whose section is a Kummer surface. But the enveloping cone of $(a\alpha)(ay)^3 = 0$ from α is

$$3[(a\alpha)^2(ay)^2]^2 - 4(a\alpha)^3(ay)\cdot(a\alpha)(ay)^3 = 0,$$

which, since $(a\alpha)^4 = 0$, reduces to $K = 0$. Hence we have shown that

(63) *If α is any point on J_4 the polar cubic of α as to J_4 is a Segre cubic spread with ten nodes on J_{10} where M_2^{18} cuts S_4, whose enveloping cone from α is the Kummer cone K. The family of ∞^3 doubly covered N_2^6's in S_4 is obtained by taking the complete intersection of the polar cubic and the polar quadric of α as to J_4 as α runs over J_4. The N_2^6 determined by α has a node at α and its cone of projection from α is K. The family of ∞^3 doubly covered surfaces W_2^4 in S_3 is the system of ∞^3 Weddle surfaces J which lie in the linear system on W_{40}.*

The last statement can be proved as follows: Let y, z be a general point on M_2^{18} for which therefore not all the y's nor all the z's can vanish. This point satisfies the last four equations (51) and we should ordinarily expect to be able to solve them for the ratios of the y's in terms of α, z. The matrix of the system is

$$(64) \qquad \left\| \begin{array}{lllll} -\alpha_1 z_1 & \alpha_0 z_1 & \alpha_3 z_4 - \alpha_4 z_3 & \alpha_4 z_2 - \alpha_2 z_4 & \alpha_2 z_3 - \alpha_3 z_2 \\ -\alpha_2 z_2 & -\alpha_3 z_4 - \alpha_4 z_3 & \alpha_0 z_2 & \alpha_1 z_4 + \alpha_4 z_1 & \alpha_1 z_3 - \alpha_3 z_1 \\ -\alpha_3 z_3 & -\alpha_2 z_4 - \alpha_4 z_2 & \alpha_1 z_4 - \alpha_4 z_1 & \alpha_0 z_3 & \alpha_1 z_2 + \alpha_2 z_1 \\ -\alpha_4 z_4 & -\alpha_2 z_3 - \alpha_3 z_2 & \alpha_1 z_3 + \alpha_3 z_1 & \alpha_1 z_2 - \alpha_2 z_1 & \alpha_0 z_4 \end{array} \right\|.$$

But the same system arranged in terms of z has the determinant K in (8) and it is satisfied by $y = \alpha$ whatever be z. Since, for the general point y, z of M_2^{18}, $y \neq \alpha$ we must have as the solution of (64) $y_i = \alpha_i f(z)$ where $f(z)$ vanishes for the general point of M_2^{18}, i. e., for the general point of W_2^4 as well. By comparing the determinant value of y_0 in (64) with J in (57) we see that

$$(65) \qquad y_i = \alpha_i \frac{J}{6},$$

whence $f(z)$ is J.

Let us now seek a parametric equation for the manifold M_2^{18} which is determined by a point α on J_4. Let $P(\bar{y}, \bar{z})$ be the general point on M_2^{18}, $P(\bar{y})$

* Cf. Bertini, *Introduzione alla geometria projettiva degli iperspazi*, pp. 176–87.

its projection on S_4, and $P(\bar{z})$ its projection on S_3. If y is an arbitrary point on the cone K determined by α, the line $\overline{\alpha y}$ meets the quadric $(a\alpha)^2(ay)^2 = 0$ in a single point other than α, which is the required point $P(\bar{y})$ on N_2^6. If $\bar{y} = y + \lambda\alpha$ we have on substituting in $(a\alpha)^2(ay)^2 = 0$ and using $(a\alpha)^4 = 0$ that $(a\alpha)^2(ay)^2 + 2\lambda(a\alpha)^3(ay) = 0$, whence

$$\bar{y}_i = 2y_i(a\alpha)^3(ay) - \alpha_i\cdot(a\alpha)^2(ay)^2. \tag{66}$$

The $\bar{z}_1, \cdots, \bar{z}_4$ were determined from 4 linear equations with the symmetric determinant K. If then we denote this determinant bordered with variables w by $K(w)$

$$K(w)_{K=0} = b_{ik}\, w_i\, w_k = (zw)^2. \tag{67}$$

Hence we have for any point y on K a unique point z whose coördinates are determined to within sign by $K(w) = (zw)^2$, or for which the products $z_i z_k$ are uniquely determined as minor determinants of K. For the required point $P(\bar{z})$ we must have $\bar{z}_i = \mu z_i$ and we have to determine this two-valued function μ. Either z or $\bar{z}$ will with $\bar{y}$ satisfy the last four equations (51) and we merely have to determine μ so that the first five are (or any one of them is) satisfied. Let us take then the first equation and set $\bar{y}_i$ equal to its value in (66) and $\bar{z}_i$ equal to μz_i where $z_i z_k$ is obtained from (67). If we note that in (61) when $(a\alpha)^4 = 0$ and $K = 0$ then

$$[(a\alpha)^2(ay)^2]^2 = 4/3\,(a\alpha)^3(ay)\cdot(a\alpha)(ay)^3$$

we find that $\alpha_0\,\bar{y}_1^2 + 2\sum_4 \alpha_1\,\bar{y}_1^2$ becomes

$$4(a\alpha)^3(ay)\{(\alpha_0\, y_0^2 + 2\sum_4 \alpha_1\, y_1^2)\cdot(a\alpha)^3(ay)$$
$$-(\alpha_0^2\, y_0 + 2\sum_4 \alpha_1^2\, y_1)\cdot(a\alpha)^2(ay)^2 + \tfrac{1}{3}(\alpha_0^3 + 2\sum_4 \alpha_1^3)\cdot(a\alpha)(ay)^3\}.$$

On the other hand we find that $\sum_4 - 2\alpha_1\,\bar{z}_1^2$ becomes

$$2\mu^2\{\sum_4 \alpha_1[\pi_{02}\,\pi_{03}\,\pi_{04} - \sum_3 \pi_{02}\,\pi_{12}^2 + 2\pi_{12}\,\pi_{13}\,\pi_{14}]\}$$
$$= 2\mu^2\{-4\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\, y_0^3 + 3\alpha_0\, y_0^2 \sum_4 \alpha_2\,\alpha_3\,\alpha_4\, y_1 - 2\alpha_0^2\, y_0 \sum_6 \alpha_1\,\alpha_2\, y_3\, y_4$$
$$+ 2y_0 \sum_{12} \alpha_1^3\,\alpha_2\, y_2^2 - 4y_0 \sum_6 \alpha_1^2\,\alpha_2^2\, y_1\, y_2 + \alpha_0^3 \sum_4 \alpha_1\, y_2\, y_3\, y_4$$
$$- \alpha_0 \sum_{12} \alpha_1^3\, y_2^3 + \alpha_0 \sum_{12} \alpha_1^2\,\alpha_2\, y_1\, y_2^2 - 2\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4 \sum_4 y_1^3$$
$$+ 2\sum_{12} \alpha_1^2\,\alpha_3\,\alpha_4\, y_1^2\, y_2 - 2\sum_{12} \alpha_1^3\,\alpha_2\, y_1\, y_3\, y_4 + 2\sum_4 \alpha_1^4\, y_2\, y_3\, y_4\}.$$

If we substitute the values of the polars in the first brace above the coefficient of $4(a\alpha)^3(ay)$ turns out to be four times the coefficient of $2\mu^2$ whence

$$8(a\alpha)^3(ay) + \mu^2 = 0.$$

(68) *If α is any point of J_4 and if y is any point of the Kummer cone K (or*

any point of the Kummer surface obtained from an S_3 section of K) the parametric equation of the normal hyperelliptic surface M_2^{18} in S_8 in terms of the parameter y is furnished by the equations

$$(v\bar{y}) = 2(vy)\cdot(a\alpha)^3(ay) - (v\alpha)\cdot(a\alpha)^2(ay)^2,$$
$$(w\bar{z}) = \sqrt{-8(a\alpha)^3(ay)\cdot K(w)}.$$

The moduli $\alpha_0 : \alpha_1 : \cdots : \alpha_4$ satisfy the single relation $(a\alpha)^4 = 0$; the parameters $y_0 : \cdots : y_4$ satisfy the single relation $K = 0$. This parametric equation of M_2^{18} is invariant in form when the cogredient variables α, y, $\bar{y}$ and the contragredient variables v are subjected to the operations of the Burkhardt G_{25920} if at the same time the contragredient variables $\bar{z}$, w are subjected to the corresponding operations of the isomorphic Maschke G_{51840}.

It is to be noted that the above parametric equations are independent of factors of proportionality in the α's and the y's but are not independent of such a factor in the coefficients a of J_4 which is supposed to be taken with the definite numerical coefficients used throughout. [See Footnote, p. 363.]

The foregoing discussion has developed a number of *kleinian forms*, i. e., forms in the variables from S_3 and S_4 which are unaltered when these variables are transformed under the operations of the isomorphic modular groups. We shall denote such a form by $k(i;j:k;l)$ if i and j are the order and class in S_4, k and l the order and class in S_3. If the form contains more than one series of cogredient variables this will be indicated by giving in the proper space the orders in the various series. A first kleinian form can be obtained from the identical covariant (vy) in S_4 if y is replaced by the cogredient expressions (55) in z. It is

$$k_1(0;1:4;0) \doteq v_0\cdot 6z_1 z_2 z_3 z_4 + \cdots + v_4[z_4(z_1^3 + z_2^3 - z_3^3)]. \tag{69}$$

The equation $k_1 = 0$ determines for given z the corresponding point α on J_4; for given v it determines a quartic of the system on W_{40} which if v touches J_4 is the Weddle J. If we operate with k_1 on J_4 we get the form k_2 in (57),

$$k_2(3;0:4;0) = J. \tag{70}$$

The $K(w)$ of (67) is another kleinian form

$$\begin{aligned} k_3(\widehat{3,3};0:0;2) = K(w) = &- w_1^2\{(\alpha_0^3 + 2\alpha_1^3)y_2 y_3 y_4 \\ &- (y_0^3 + 2y_1^3)\alpha_2\alpha_3\alpha_4 - (\alpha_0^2 y_0 + 2\alpha_1^2 y_1)\sum_3 \alpha_2 y_3 y_4 \\ &+ (\alpha_0 y_0^2 + 2\alpha_1 y_1^2)\sum_3 y_2\alpha_3\alpha_4 + y_0\alpha_1^2\sum_3 \alpha_2 y_2^2 - \alpha_0 y_1^2 \sum_3 \alpha_2^2 y_2 \\ &+ 2\alpha_0\alpha_1 y_1 \sum_3 \alpha_2 y_2^2 - 2\alpha_1 y_0 y_1 \sum_3 \alpha_2^2 y_2 + y_0 y_1^2 \sum_3 \alpha_2^3 \\ &- \alpha_0\alpha_1^2 \sum_3 y_2^3\} + \cdots, \end{aligned} \tag{71}$$

where $\widehat{3, 3}$ indicates that k_3 is of degree 3 in the line coördinates $\widehat{\alpha y}$. For α on J_4 and y on K, $k_3 = 0$ is the equation of the node of the Weddle at the point z which corresponds to y. For α on J_4 and given w, $k_3 = 0$ is the equation of a cubic cone which touches K along a sextic 2-way cone which corresponds to the plane section of the Weddle.

If we operate (y on v) with k_3 upon the invariant $J_{0,\,4}$ in S_4 we get

$$(72)\qquad k_4(3;1:0;2) = w_1^2[18\{\alpha_0\alpha_1^2 + 2\alpha_2\alpha_3\alpha_4\}v_0 - 3\{\alpha_0^3 + \tfrac{2}{3}\sum_4 \alpha_1^3\}v_1] + \cdots.$$

If again we operate (α on v) on $J_{0,\,4}$ we get

$$(73)\qquad k_5(0;\widehat{1, 1}:0;2) = 216w_1^2\{v_0v_1' - v_1v_0'\} + \cdots.$$

By reason of the self dual character of a collineation group we should expect to find a form $k_6(\widehat{1, 1};0:2;0)$ dual to k_5 and the explicit equation of k_6 can be found as follows. Multiply the first five quadrics (51) respectively by $y_0, 2y_1, \cdots, 2y_4$ and add. Then

$$(74)\qquad (a\alpha)(ay)^3 + 2\{\sum_4 \pi_{01}z_1^2 + 2(\pi_{12}z_3z_4 + \pi_{13}z_4z_2 + \pi_{14}z_2z_3 + \pi_{32}z_1z_4 + \pi_{24}z_1z_3 + \pi_{43}z_1z_2)\} = 0.$$

Since y is a point on N_2^6, $(a\alpha)(ay)^3 = 0$ and we take k_6 to be

$$(75)\qquad k_6(\widehat{1, 1};0:2;0) = \sum_4 \pi_{01}z_1^2 + 2\pi_{12}z_3z_4 + \cdots + 2\pi_{43}z_1z_2.$$

If α is on J_4 and y on N_2^6 (or on K since y occurs only in the combinations π_{ik}), $k_6 = 0$ is a quadric on the 6 nodes of J since it was formed from quadrics (54). Also it is a quadric with a node since its discriminant is K. The coördinates $\bar{z}$ of the node are obtained from $K(w)$ and according to (68) $\bar{z}$ is the point of W_2^4 which corresponds to y on N_2^6. Hence

(76) *If M_2^{18} is determined by α on J_4 and if y and z are corresponding points on the doubly covered projections N_2^6 and W_2^4 then, for given y, $k_6 = 0$ is the quadric with node at z and on the six nodes of W_2^4.*

On account of the 2 to 1 isomorphism between the Maschke and the Burkhardt group we cannot expect to find kleinian forms in y and z alone which are linear in z. But such forms which are quadratic in z can be found. If for example we operate (v on y) with k_4 upon J_4 we get a $k_7(6;0:0;2)$ and the point equation of this quadric in w would be a form of the sort required. If two such forms, quadratic in z with coefficients containing y, are derived

Footnote. Parametric equations for the normal elliptic curves of orders 3, 4, 5 to (68) are found in Dr. B. I. Miller's dissertation; these Transactions, vol. 17 (1916), p. 259. The coördinates of a point of the curve are expressed in covariant form in terms of rational functions of the parameter of a point on a line and the radical of such a function. In the above equations the point on a line is replaced by the point on a Kummer surface.

their pencil will contain at least one and in general four quadric cones. One of these cones can be isolated by the solution of an accessory quartic equation and its node will be a "covariant point," i. e., a point z whose coördinates are functions of y and which undergoes the collineations of the Maschke group when the y's are subjected to the operations of the Burkhardt group. By the use of such a point the form problem of the Burkhardt group can be solved in terms of a solution of the form problem of the Maschke group.*

We shall however attack the Burkhardt form problem directly and in the next paragraph indicate how it can be solved in the special case when $J_4 = 0$. In § 7 this restriction will be removed by using an accessory quartic equation.

6. Solution of the special Burkhardt form problem

The form problem of the Burkhardt group when $J_4 = 0$ (referred to as the "special" Burkhardt form problem) reads as follows: Given the values of the absolute invariants $\lambda = J_{12}/J_6^2$, $\mu = J_{18}/J_6^3$, $\nu = J_{10}^3/J_6^5$ to find the ratios of the coördinates $\alpha_0, \cdots, \alpha_4$ of a point on $J_4 = 0$ for which these invariants take the assigned values. The problem has 25920 solutions since the spreads $J_{12} - \lambda J_6^2, \cdots, J_{10}^3 - \nu J_6^5 = 0$ meet $J_4 = 0$ in $4 \cdot 12 \cdot 18 \cdot 30 = 25920$ points.

We shall develop the solution of this problem under the following heads.

1°. A point α on $J_4 = 0$ determines a binary sextic (projective to the sextic similarly determined at any one of the set of points conjugate to α on J_4) which is the fundamental sextic of the hyperelliptic functions.

2°. The absolute invariants of this sextic are precisely the given λ, μ, ν. These will be identified with a known system.

3°. The equation of this sextic will be expressed in terms of these invariants by means of an accessory square root.

4°. With this explicitly given sextic, the hyperelliptic algebraic relation of genus two is determined. Assuming that the transcendental operations involved in the determination of the periods of a pair of integrals of the first kind have been effected, the coördinates $\alpha_0, \cdots, \alpha_4$ are expressed by means of certain known series.

1°. From the equation (51) of the Kummer cone K determined at a point α of J_4 we see that the S_3, $(a\alpha)^3(ay) = 0$, and the quadric, $(a\alpha)^2(ay)^2 = 0$, meet in a trope of K. The six double lines of K on this trope are on

$$(a\alpha)(ay)^3 = 0.$$

Hence the cubic, quadratic, and linear polars of α as to J_4 meet in six lines of a quadric cone in S_3, the meet of $(a\alpha)^3(ay) = 0$ and $(a\alpha)^2(ay)^2 = 0$ which

* As to the difficulty of the reduction here indicated cf. BIII, p. 339*. The use of kleinian forms to determine covariant points is illustrated in the article: Coble, *Reduction of the sextic equation to the Valentiner form problem*, Mathematische Annalen, vol. 70 (1911), p. 337.

touch at α. The binary sextic

$$S = s_0 s^6 + 6 s_1 s^5 + 15 s_2 s^4 + \cdots + s_6$$

determined by these six lines on the cone is the fundamental sextic associated with K and therefore with the hyperelliptic theta functions used here.

2°. We have seen that the polar cubic spread $C \equiv (a\alpha)(ay)^3 = 0$, is on α of J_4 and has ten nodes on J_{10}. Through α there will be six lines which lie entirely on C, the six lines of 1°. The lines through α which touch C again are the lines of K. This spread C can be mapped from an S_3 by means of quadrics $q_a, \cdots, q_f$ on five points $p_1, \cdots, p_5$ of S_3 where

$$C \equiv q_a^3 + \cdots + q_f^3, \qquad q_a + \cdots + q_f \equiv 0. \tag{77}$$

If a sixth point p maps on the point $\alpha = \bar{q}_a, \cdots, \bar{q}_f$ of C then the section of C by the polar quadric $q_a^2 \bar{q}_a + \cdots + q_f^2 \bar{q}_f = 0$ is the map of a Weddle quartic in S_3 with nodes at $p_1, \cdots, p_5, p$. The section of C by the polar space $q_a \bar{q}_a^2 + \cdots + q_f \bar{q}_f^2 = 0$ is the map of a quadric cone in S_3 with node at p and on $p_1, \cdots, p_5$. A convenient analytic representation for this is found in C1, § 1. The quadric cone meets the Weddle in the 5 lines from p to $p_1, \cdots, p_5$ and in the cubic curve on the six points. These six curves through p map into the six lines of 1° on α in S_4 and the directions on the quadric cone at p, which are the same as those on the Weddle at p, map into directions on $(a\alpha)^2(ay)^2 = (a\alpha)^3(ay) = 0$ about α. Hence the sextic S in S_4 is projective to the sextic in S_3 determined on the quadric cone with node at p by the six curves mentioned and therefore to the sextic determined on the cubic curve by the points $p_1, \cdots, p_5, p$ (or to the sextic $y_0, \cdots, y_4, \infty$ of C1, § 1). The invariants of this sextic have been determined as symmetric functions of the q's (C1, (7), (10)). Thus the problem before us is: Given a spread C which can be linearly transformed into the special form (77), and a point α on it which is transformed into $\bar{q}_a, \cdots, \bar{q}_f$, ($a$) to determine those covariants of C which are transformed into the elementary symmetric functions $q_2, \cdots, q_6$ of the q's, and (b) to find their values for the particular point $\alpha = q_a, \cdots, \bar{q}_f$.

2° (a). We have first of all $C = 3q_3$. We shall denote the discriminant of the polar quadric of C by H and the dual equation of this quadric in variables $r_a, \cdots, r_f$ by B. If then we take account of the supernumerary coördinates

$$B = \begin{vmatrix} q_a & 0 & \cdots & 0 & 1 & r_a \\ 0 & q_b & \cdots & 0 & 1 & r_b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & q_f & 1 & r_f \\ 1 & 1 & \cdots & 1 & 0 & 0 \\ r_a & r_b & \cdots & r_f & 0 & 0 \end{vmatrix}.$$

Thus

$$H = -q_5$$

and

$$B = \sum_{15} q_a q_b q_c q_d (r_e - r_f)^2.$$

If we operate with B on H we get

$$[B, H] = 2 \sum_{60} q_a^2 q_b^2 q_c^2 q_d = 2(q_3 q_4 - 3q_2 q_5).$$

If now we allow for a factor σ in the coefficients of C we have

$$C = 3\sigma q_3, \qquad H = -\sigma^5 q_5,$$

$$[B, H] = 2\sigma^9 (q_3 q_4 - 3q_2 q_5) = 2(\sigma^8 q_4 C/3 + 3H \cdot \sigma^4 q_2).$$

From the latter syzygy we can determine $\sigma^8 q_4$ and $\sigma^4 q_2$.

If we operate with B^2 on H we get $-72 q_6 q_3$, whence from

$$[B^2, H] = -72\sigma^{13} q_6 q_3 = -24\sigma^{12} q_6 \cdot C$$

we can determine $\sigma^{12} q_6$.

The term q_5 derived from H alone does not acquire the proper power of σ so it must be modified. The cubic spread in S_4 has an invariant A of degree 10 in the coefficients which can be obtained by forming according to the Clebsch principle from the invariant i_8 of the cubic surface in S_3 a contravariant of C in S_4 of degree 8 and class 6 and by operating with this contravariant on C^2. For the above spread the invariant A takes a purely numerical value which we shall denote by a. Instead of using H alone we shall use

$$\frac{1}{\sqrt{a}} \sqrt{A} H = -\sigma^{10} q_5.$$

2° (*b*). We now identify the cubic spread C with the polar spread $\bar{C}$ of α on J_4, i. e.,

$$\bar{C} = \alpha_0 [y_0^3 + 2 \sum_4 y_1^3] + 6 \sum_4 \alpha_1 (y_0 y_1^2 + 2 y_2 y_3 y_4)$$

$$\equiv \sigma (q_\alpha + \cdots + q_{\gamma'}^3) \equiv C.$$

We calculate the covariants $\bar{H}$ and $\bar{B}$ and the invariant $\bar{A}$ of $\bar{C}$ and equate them to the corresponding forms of C taking account of the determinant δ of the transformation, and we identify similarly the results of corresponding operations. We shall want however not the covariants themselves but rather the values which they take at the point $y = \alpha$ so that this substitution can be made after the operations have been completed. Since also we have merely to identify certain numerical coefficients we shall use the special case when $\alpha_3 = \alpha_4 = 0$ and set

$$\alpha_0^3 = t_0, \qquad \alpha_1^3 = t_1, \qquad \alpha_2^3 = t_2.$$

Then Burkhardt's invariants become

$$\begin{aligned}
J_4(\alpha) &= \alpha_0[t_0 + 8(t_1 + t_2)],\\
J_6(\alpha) &= t_0^2 - 20t_0(t_1 + t_2) - 8(t_1^2 + t_2^2) + 80t_1 t_2,\\
J_{10}(\alpha) &= \alpha_0 t_1 t_2[-t_0 + (t_1 + t_2)],\\
J_{12}(\alpha) &= t_1 t_2[5t_0^2 - t_0(t_1 + t_2) - 4(t_1^2 + t_2^2) + 16t_1 t_2],\\
J_{18}(\alpha) &= t_1^2 t_2^2[-t_0^2 + 2t_0(t_1 + t_2) - (t_1^2 + t_2^2) + 2t_1 t_2].
\end{aligned}$$

By a straightforward calculation we find that

$$\begin{aligned}
\overline{B} = 16\cdot 4\{&4t_1 t_2 \alpha_0^2 u_0^2 - 4t_1 t_2 \alpha_0 u_0(\alpha_1 u_1 + \alpha_2 u_2) + t_2(t_0 - t_2)\alpha_1^2 u_1^2\\
&+ t_1(t_0 - t_1)\alpha_2^2 u_2^2 + 2t_1 t_2 \alpha_1 \alpha_2 u_1 u_2 + 2\alpha_0 \alpha_1^2 \alpha_2^2(t_0 - t_1 - t_2) u_3 u_4\}.\\
\overline{H} = 16\{&-\alpha_0 \alpha_1 \alpha_2 y_0^3(\alpha_1 y_2 + \alpha_2 y_1)^2 - y_0^2 \alpha_0^2(\alpha_1 y_2 + \alpha_2 y_1)^3 + y_0[2t_2 \alpha_1^2 y_4\\
&+ (4t_1 - t_0)\alpha_2^2 y_1^3 y_2 + 2\alpha_1 \alpha_2(-t_0 + t_1 + t_2) y_1^2 y_2^2 + \alpha_1^2(4t_2 - t_0) y_1 y_2^3\\
&+ 2\alpha_2^2 t_1 y_2^4] + 2\alpha_0 \alpha_1^2 \alpha_2^2 y_1^4 y_2 + 2\alpha_0 \alpha_2(t_2 + 2t_1) y_1^3 y_2^2\\
&+ 2\alpha_0 \alpha_1(t_1 + 2t_2) y_1^2 y_2^3 + 2\alpha_0 \alpha_1^2 \alpha_2^2 y_1 y_2^4 + y_3 y_4[y_0^3 t_0 \alpha_1 \alpha_2\\
&+ y_0^2 \alpha_0(t_0 + 2t_1 + 2t_2)(\alpha_1 y_2 + \alpha_2 y_1) + y_0 \alpha_0^2(t_0 + 2t_1 + 2t_2) y_1 y_2\\
&- 4y_1^3 t_1 \alpha_1 \alpha_2 - 2y_1^2 y_2 \alpha_1^2(t_0 + 2t_1 - 4t_2) - 2y_1 y_2^2 \alpha_2^2(t_0 - 4t_1 + 2t_2)\\
&- 4y_2^3 t_2 \alpha_1 \alpha_2] + \text{terms in } y_3^3 \text{ and } y_4^3\}.
\end{aligned}$$

We now find at once that $\overline{H}_{y=\alpha} = 16^2 J_{10}(\alpha)$, and $\overline{C}_{y=\alpha} = J_4(\alpha)$. If we operate with $\overline{B}$ on $\overline{H}$ and set $y = \alpha$ we have

$$\begin{aligned}
[\overline{B}, \overline{H}]_{y=\alpha} &= 16^3 \alpha_0 t_1 t_2\{2t_0^3 - 6t_0^2(t_1 + t_2) + 6t_0(t_1^2 + t_2^2) + 90t_0 t_1 t_2\\
&\qquad - 2(t_1^3 + t_2^3) - 30t_1 t_2(t_1 + t_2)\}\\
&= 16^2\cdot 4\{J_4(\alpha)\cdot J_{12}(\alpha) - 3J_6(\alpha)\cdot J_{10}(\alpha)\}.
\end{aligned}$$

If we operate with $\overline{B}^2$ on $\overline{H}$ and set $y = \alpha$ we get

$$\begin{aligned}
[\overline{B}^2, \overline{H}]_{y=\alpha} &= 16^4\cdot 8\cdot 3\alpha_0 t_1^2 t_2^2\{t_0^3 + 6t_0^2(t_1 + t_2) - 15t_0(t_1^2 + t_2^2)\\
&\qquad - 34t_0 t_1 t_2 + 8(t_1^3 + t_2^3) - 8t_1 t_2(t_1 + t_2)\}\\
&= -16^4\cdot 24 J_4(\alpha)\cdot J_{18}(\alpha).
\end{aligned}$$

The invariant $\bar{A}$ of $\overline{C}$ must be a numerical multiple of $J_{10}(\alpha)$. For its vanishing implies that the polar cubic can not be thrown into the form (77) and this is true only when the hyperelliptic spreads degenerate due to the vanishing of the discriminant of the underlying sextic S. But this discriminant is represented on $J_4(\alpha)$ by $J_{10}(\alpha)$.* We shall assume that

$$\frac{\bar{A}}{a} = \lambda J_{10}(\alpha)$$

and leave the so defined numerical constant λ undetermined.

* Cf. BIII, p. 337 (3) and p. 331 (2).

We now have after setting $\sigma = 1$ in the earlier formulæ the following relations

$$\bar{C}_{y=\alpha} = J_4(\alpha) = 3q_3,$$

$$\sqrt{\frac{\bar{A}}{a}}\,\bar{H}_{y=\alpha} = \sqrt{\lambda J_{10}(\alpha)}\cdot 16^2 J_{10}(\alpha) = -\,\delta^5 q_5,$$

$$\bar{H}_{y=\alpha} = 16^2 J_{10}(\alpha) = -\,\delta^2 q_5,$$

$$[\bar{B}, \bar{H}]_{y=\alpha} = 16^2\cdot 4\{J_4(\alpha)\cdot J_{12}(\alpha) - 3J_6(\alpha)\cdot J_{10}(\alpha)\} = 2\delta^4(q_3 q_4 - 3q_2 q_5)$$

$$[\bar{B}^2, \bar{H}]_{y=\alpha} = -\,16^4\cdot 24 J_4(\alpha)\cdot J_{18}(\alpha) = -\,24\cdot 3\delta^6 q_3 q_6.$$

Hence

$$\begin{aligned} \delta^2 q_2 &= -\,2J_6,\\ \delta^4 q_4 &= 16\cdot 6J_{12},\\ \delta^6 q_6 &= 16^4 J_{18},\\ \delta^5 q_5 &= -\,16^2\sqrt{\lambda J_{10}^3}. \end{aligned} \tag{78}$$

From the two values of q_5 above we have $\delta = (\lambda J_{10}(\alpha))^{1/6}$. If we account for it by introducing a factor $1/\delta$ in the roots of the fundamental sextic S we have finally

(79) *The resolvent sextic* Σ (C1, p. 317 (9)) *of the fundamental sextic S is*

$$Q^6 - 2J_6 Q^4 + 16^2\cdot 6J_{12} Q^2 + 16^2\sqrt{\lambda J_{10}^3}\, Q + 16^4 J_{18} = 0;$$

and the invariants A, B, C, Δ (notation of C1, p. 317 (7)) *of S are given by the equations*

$$-\,6J_6 = 5A, \qquad 16^2\cdot 4\cdot 3^3 J_{12} = 5(6A^2 - 5^2 B),$$

$$16^4\cdot 2\cdot 3^4 J_{18} = 5(-\,6A^3 + 3\cdot 5^2 A\cdot B + 2\cdot 5^3 C),$$

$$16^4\cdot 3^5 \lambda J_{10}^3 = \Delta = \Pi(s_i - s_k)^2.*$$

3°. In order to exhibit an explicit sextic S we can make use of a typical representation of S.† The sextic has three quadratic covariants connected by an identical relation, $K_2 = 0$, of the second order whose coefficients are rational in A, B, C, Δ. The sextic S is then expressed as a form $S = K_3 = 0$

* I had first attempted to calculate these invariants by the following method. In BIII, p. 331, the sextic S is taken with one root at ∞ and the sum of the others zero. Then the coefficients $g_2, \cdots, g_6$ are given in terms of $[H], \cdots, [\psi]$ which on p. 331 (2) are expressed by means of $(f_{12}), \cdots, (f_{40})$. Then in § 73, p. 336, the $J_6(\alpha)$ and $J_{12}(\alpha)$ for $J_4(\alpha) = 0$ are calculated in terms of the $(f_{12}), \cdots, (f_{40})$. As expected $J_6(\alpha)$ furnished the self apolarity invariant of the sextic S but neither Burkhardt's value [(14), p. 339] of $J_{12}(\alpha)$ nor the different one which I obtained would furnish a second invariant of the sextic. This discrepancy and the one noted above may be due to a single error in the expression for J_{12}. An advantage of the method carried through here is that the invariants are identified at once with a known complete system.

† Gordan, *Invariantentheorie*, p. 302.

of order three in l, m, n with similar coefficients. Thus the sextic appears as the six points cut out on the conic $K_2 = 0$ by the cubic $K_3 = 0$. In order to introduce a parameter on K_2 an accessory square root is required. This may be for example the square root of the discriminant of the quadratic in m, n obtained by setting $l = 0$ in K_2. When a point on the conic K_2 is obtained in terms of a parameter s then the cubic $K_3 = 0$ determines the sextic S. Thus we have

$$K_2 \equiv \begin{vmatrix} A_{ll} & A_{lm} & A_{ln} & l \\ A_{ml} & A_{mm} & A_{mn} & m \\ A_{nl} & A_{nm} & A_{nn} & n \\ l & m & n & 0 \end{vmatrix} = a_{11}\, l^2 + \cdots + 2a_{23}\, mn.$$

If we set

$$r = \sqrt{a_{23}^2 - a_{22}\, a_{33}} = \sqrt{-\,2R^2 A_{ll}}; \qquad f_2 = \frac{\partial K_2}{\partial m}, \qquad f_3 = \frac{\partial K_2}{\partial n},$$

then we can solve the system of equations

$$\begin{aligned} 2R^2\, l &= s, \\ (-\,a_{23} + r) f_2 + a_{22} f_3 &= a_{22}\, s^2, \\ (-\,a_{23} - r) f_2 + a_{22} f_3 &= 1, \end{aligned}$$

for l, m, n as quadratics in s, and thereby obtain from K_3 the sextic S. The necessary formulas for explicit expressions in A, B, C, Δ are given by Gordan, pp. 288–90.

4°. From the algebraic relation $t = \sqrt{S}$ we assume that the periods $\omega_{i,\,1}$, $\omega_{i,\,2}$ $(i = 1, \cdots, 4)$ of a pair of integrals of the first kind have been calculated.* From the known series for $X_{\alpha\beta}$† these quantities are calculated for $(u) \equiv 0$. Then formulæ (48) furnish the coördinates of a point y on J_{10} whose polar quadric as to J_4 has a double point at the required point α on J_4.

7. Solution of the general Burkhardt form problem

We shall state the Burkhardt form problem as follows:‡ Given the numerical values of J_4, J_6, J_{10}, J_{12}, J_{18} to find the ratios of the coördinates y for which these forms take the given values. If the ratios are found the actual coördinates can be obtained to within sign by using the numerical value of J_6/J_4 to determine the square of the factor of proportionality. The solution required can be given in terms of the solution of the special problem considered in § 6 by conventional methods.§ We shall employ the phrase "de-

* Various methods for this are reviewed in BI, § 45, p. 277. That of Wiltheiss, Mathematische Annalen, vol. 31, p. 141, would be more in line with the above account since it implies no separation of the roots of S.

† BII, p. 171 (17).

‡ Cf. BII, §§ 51–2, p. 214.

§ Cf. Klein, *Ikosaeder*, II, 5, § 2, p. 241.

terminate polynomial J_i" to indicate a polynomial in $J_4, J_6, \cdots, J_{18}$ with numerical coefficients and of degree i in y. These numerical coefficients can be calculated once for all (which we assume has been done in the cases below) and the polynomial takes a numerical value when the form problem is given.

Let y be one of the required points. It can be given by the equation $J_{1,1} = 0$. Then $\lambda J_{1,1} + J_{7,1} = 0$ is for variable λ a line "covariant" with y, i. e., the relation between y and the line is unaltered under the transformations of G_{25920} because of the invariance of $J_{1,1}$ and $J_{7,1}$. This line meets J_4 in four points whose parameters λ are the roots of the equation

$$(80) \qquad J_4\lambda^4 + J'_{10}\lambda^3 + J'_{16}\lambda^2 + J'_{22}\lambda + J'_{28} = 0.$$

In this equation J'_i is a determinate polynomial. A root λ of (80) is an irrational invariant of G_{25920} of degree 6. If one such root—an accessory irrationality—has been adjoined, the point $J_{1,1}\lambda + J_{7,1} = 0$ is a point covariant with y and on J_4.

We shall denote by $\bar{J}_6, \bar{J}_{10}, \bar{J}_{12}, \bar{J}_{18}$ the values which $J_6, \cdots, J_{18}$ take for this covariant point. Then

$$\bar{J}_\rho = J_\rho\lambda^\rho + J'_{\rho+6}\lambda^{\rho-1} + \cdots + J'_{\rho+6k}\lambda^{\rho-k} + \cdots + J'_{7\rho},$$

where $J'_{\rho+6k}$ is a determinate polynomial. Since by means of (80) $J_4^{\rho-3}\lambda^\rho$ can be expressed linearly in terms of $\lambda^3, \lambda^2, \lambda, 1$ with determinate polynomial coefficients we can set

$$(81) \qquad J_4^{\rho-3}\bar{J}_\rho = J_{\rho_0}\lambda^3 + J_{\rho_1}\lambda^2 + J_{\rho_2}\lambda + J_{\rho_3} \quad (\rho = 6, 10, 12, 18),$$

where J_{ρ_i} is a determinate polynomial, for which $\rho_i = 11\rho + 6i - 30$.

We shall denote by $\bar{J}_{1,1}, \bar{J}_{7,1}, \bar{J}_{9,1}, \bar{J}_{13,1}, \bar{J}_{15,1}$ the values which $J_{1,1}, \cdots, J_{15,1}$ take when for y there is substituted the covariant point

$$\lambda J_{1,1} + J_{7,1} = 0$$

on J_4. Then

$$\bar{J}_{\sigma,1} = J_{\sigma,1}\lambda^\sigma + J'_{\sigma+6,1}\lambda^{\sigma-1} + \cdots + J'_{\sigma+6k,1}\lambda^{\sigma-k} + \cdots + J'_{7\sigma,1},$$

where $J'_{\sigma+6k,1}$ can be expressed linearly in terms of $J_{1,1}, \cdots, J_{15,1}$ with determinate polynomial coefficients. Again multiplying by $J_4^{\sigma-3}$ to reduce the exponents of λ to 3 or less we can set

$$(82) \qquad \begin{aligned} J_4^{\sigma-3}\bar{J}_{\sigma,1} = {} & [J_{\sigma_{0,1}}\lambda^3 + J_{\sigma_{1,1}}\lambda^2 + J_{\sigma_{2,1}}\lambda + J_{\sigma_{3,1}}]J_{1,1} \\ & + [J_{\sigma_{0,7}}\lambda^3 + J_{\sigma_{1,7}}\lambda^2 + J_{\sigma_{2,7}}\lambda + J_{\sigma_{3,7}}]J_{7,1} \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & + [J_{\sigma_{0,15}}\lambda^3 + J_{\sigma_{1,15}}\lambda^2 + J_{\sigma_{2,15}}\lambda + J_{\sigma_{3,15}}]J_{15,1}, \\ & (\sigma = 1, 7, 9, 13, 15), \end{aligned}$$

where $J_{\sigma_{ij}}$ is a determinate polynomial and $\sigma_{ij} = 11\sigma + 6i - j - 30$. All the determinate polynomials that occur above are of even degree so that J_{45} can not appear. Of course if any of the degrees indicated are negative the correspondiag terms do not appear.

To solve the given problem we first find from the numerical values of $J_4, \cdots, J_{18}$ the numerical values of the determinate polynomials in (80), (81), (82). Equation (80) is then solved and a value of the accessory irrationality λ obtained. With this value of λ and the given value of J_4 we find from (81) the values of $\bar{J}_\rho$. These are the known quantities in the special Burkhardt form problem and we find as in § 6 the covariant point α on J_4 such that $(v\alpha) = \lambda J_{1,1} + J_{7,1}$. The coördinates of α can be determined to within sign by the numerical value of $\bar{J}_{12}/\bar{J}_{10}$. With these coördinates α the linear forms $\bar{J}_{\sigma,1}$ of (82) can be determined to within sign. Since now the left members of (82) are completely known to within a change of sign throughout and the coefficients of $J_{\sigma,1}$ in the right members also are known we can solve the five linear equations for $J_{1,1}$ and therefore determine to within sign the coefficients $y_0, \cdots, y_4$ of $v_0, \cdots, v_4$.

With this the above sketch of the processes involved in the determination of the lines on a cubic surface is complete. In addition to adjoining the square root of the discriminant of the surface we have introduced an accessory square root in § 6 and an accessory quartic irrationality in this paragraph. These are to be compared with the two accessory square roots required by the method of Klein which is based on the form problem of the z's. The essential difference between the problem of the y's and the problem of the z's is that the latter implies an isolation of a root of the fundamental sextic* while the former does not. Much depends also on the way in which the transcendental operations are carried out. If for example we set the problem: Given the numerical values of $J_6, \cdots, J_{18}$ when $J_4 = 0$ to find the ratios of the coördinates of the $10 \cdot 25920$ points on J_{10} which are the nodes of the polar cubics of the 25920 points on J_4; then the algebraic adjunction of a solution of the special form problem would merely reduce the problem to the solution of an equation of degree 10 with a sextic resolvent. But if we proceed to effect the solution of the special form problem as in § 6 by the adjunction of an accessory square root then after the periods of a pair of integrals have been obtained the required points are found by substituting the 10 even half periods (including the zero half period in the transcendental expressions for the y's). We may note also that the closing remarks in § 5 indicate that a quartic irrationality may be unavoidable in effecting the solution of the problem of the y's in terms of the problem of the z's.

* Cf. BIII, §§ 66, 68, p. 327.

8. Comparison with the quintic equation

Many of the ideas in this series of articles had their origin in an earlier paper entitled " An application of the form problems associated with certain Cremona groups to the solution of equations of higher degree."* This paper contained a detailed application to the quintic and it may be of interest to trace the striking analogy between the problem of the quintic as there set forth and the problem of the determination of the lines on a cubic surface as presented in this article.

The quintic with ordered roots determined an ordered P_5^2, four of whose points were taken at a base in S_2, and the coördinates x, y, u† of the remaining point determined in turn the ordered quintic. The permutations of the roots of the quintic led to the operations of a Cremona $G_{5!}$ in x, y, u whose invariants were the invariants of the quintic itself. This corresponds to the content of our present § 1 except that here the cubic surface determines 72 ordered P_6^2's and the Cremona group is the extended group $G_{6,2}$ of order $72 \cdot 6!$ rather than $G_{6!}$. The transition from the quintic equation to the solution of the form problem of $G_{5!}$ was accomplished by a typical representation analogous to the process used in § 2. The simplest linear system of irrational invariants of the quintic of dimension 5 divided under the operations of the invariant even subgroup $G_{\frac{1}{2}5!}$ into two skew linear systems of dimension 2 which experienced under $G_{\frac{1}{2}5!}$ the linear transformations of Klein's contragredient groups of the A's and of the A''s. Precisely similar facts concerning $\Gamma_{6,2}$ appear in (36) and (40) of § 3. The solution of the form problem of $G_{\frac{1}{2}5!}$ was accomplished by the use of invariants linear in the A''s—the same device as is employed in § 4. Here the analogy ends since the solution of the problem of the A's given in the earlier paper was effected by a special method.

If however we take into account the results of Miss Miller (loc. cit., pp. 278–83) which furnish the analog of § 5, there would seem to be little doubt that a binary quartic could be attached to each point of the conic invariant under the group of the A's—a quartic projective to that which determines the elliptic quintic in S_4. Then developments (in which J_4 is replaced by the invariant conic) precisely parallel to those of §§ 6, 7 could be made and the analogy between the two given problems would persist throughout.

Baltimore,
July 28, 1916

* Coble, these Transactions, vol. 9 (1908), p. 396.

† As a matter of fact the use of a superfluous coördinate was more advantageous in that certain results of Clebsch could be utilized.

ON THE SECOND DERIVATIVES OF THE EXTREMAL-INTEGRAL FOR THE INTEGRAL $\int F(y;\, y')\, dt$*

BY

ARNOLD DRESDEN

Introduction

It is the purpose of this paper to obtain formulas for the second derivatives of the extremal-integral arising in the theory of the integral

$$\int F(y; y')\, dt, \tag{1}$$

where y, y' are symbols for the n-partite numbers $(y_1, \cdots, y_n)$ and $(y'_1, \cdots, y'_n)$, the $'$ denoting differentiation with respect to the parameter t, expressing these derivatives in terms of "normal solutions" of the Jacobi system of differential equations, introduced into the literature by Professor Bliss.† The matrix notation used in Bliss's paper will be followed here in as far as it is feasible to do so.

The methods used in this paper are analogous to those of an earlier paper;‡ for this reason only such details as are essential will be developed here. In Section 1 some preliminary facts are put together; in Section 2 the desired formulas are derived and discussed.

1. Preliminary developments

Using the arc length s as a parameter, we denote by

$$y = y(s - s_1;\, y_1,\, y'_1) \tag{2}$$

the solutions of Euler's differential equations for our problem, which satisfy the initial conditions

$$y(0; y_1, y'_1) = y_1, \qquad y'(0; y_1, y'_1) = y'_1,$$

where s_1, $y_1(= y_{11}, \cdots, y_{1n})$ and $y'_1(= y'_{11}, \cdots, y'_{1n})$ are arbitrary constants of 1, n, and n dimensions respectively, subject only to the condition

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† Bliss, these Transactions, vol. 17 (1916), p. 195.

‡ Dresden, these Transactions, vol. 17 (1916), p. 425.

$\sum_{i=1}^{i=n} y_{1i}'^2 = 1$. The existence of such a solution follows from the general existence theorems.* Supposing that we have found an extremal passing through $A_1(a_1)$ and $A_2(a_2)$, not containing a pair of conjugate points, we can construct an extremal through the points $P_1(y_1)$ and $P_2(y_2)$ taken sufficiently near to A_1 and A_2 respectively. We find†

$$(3) \qquad y = y(s - s_1; y_1, \theta(y_1, y_2)) \equiv y(s), \qquad s_1 \leqq s \leqq s_2,$$

where $\theta(y_1, y_2)$ and $s_2(y_1, y_2)$ are determined so as to satisfy the equations

$$(4) \qquad y(s_2 - s_1; y_1, \theta) = y_2, \qquad \sum_i \theta_i^2 = 1.$$

The functions $y(s)$‡ defined by equations (3) satisfy the following initial conditions:

$$(5) \qquad y(s_1) = y_1, \qquad y(s_2) = y_2, \qquad y'(s_1) = \theta(y_1, y_2).$$

If we write ϕ for $y'(s_2)$, we can obtain a second form for the equation of the extremal $P_1 P_2$, viz.:

$$(3a) \qquad y = y(s - s_2; y_2, \phi) \equiv \bar{y}(s), \qquad s_1 \leqq s \leqq s_2;$$

the functions $\bar{y}(s)$ satisfy the conditions

$$(5a) \qquad \bar{y}(s_1) = y_1, \qquad \bar{y}(s_2) = y_2, \qquad \bar{y}'(s_1) = \theta, \qquad \bar{y}'(s_2) = \phi,$$

while ϕ and s_1 satisfy the equations

$$(4a) \qquad y(s_1 - s_2; y_2, \phi) = y_1, \qquad \sum_i \phi_i^2 = 1.$$

Along the extremal $P_1 P_2$ taken in the forms (3) or (3a), we compute the integral (1). The integral obtained in this way, known as the extremal-integral connected with the integral (1), is a function of y_1 and y_2; we will denote it by $\Gamma(y_1, y_2)$:

$$\Gamma(y_1, y_2) = \int_{s_1}^{s_2} F(y(s), y'(s))\, ds = \int_{s_1}^{s_2} F(\bar{y}(s), \bar{y}'(s))\, ds.$$

If we use subscript notation for the derivatives of the function Γ

$$(\Gamma_{1i} = \partial\Gamma/\partial y_{1i}, \quad \text{etc.}, \qquad \Gamma_{1i,\,2j} = \partial^2 \Gamma/\partial y_{1i}\, \partial y_{2j}, \quad \text{etc.}),$$

* Bolza, *Vorlesungen über Variationsrechnung*, pp. 168–188.

† Bolza, loc. cit., p. 597.

‡ Whenever this can be done without sacrificing clearness, we shall write $y(s)$ in place of $y(s - s_1; y_1, \theta(y_1, y_2))$, and $F(s)$ for $F(y(s); y'(s))$; $\bar{y}(s)$ and $\bar{F}(s)$ will be used with analogous meanings.

we find*

(6) $\Gamma_{1j} = - F_{n+j}(y_1; \theta) = - F_{n+j}(s_1), \qquad \Gamma_{2j} = F_{n+j}(y_2; \phi) = F_{n+j}(s_2).$

The second derivatives of the function F taken along the extremal $P_1 P_2$ will be denoted by the symbols, used by Bolza,† viz.,

$$\frac{\partial^2 F}{\partial y_i \partial y_k} = P_{ik}, \qquad \frac{\partial^2 F}{\partial y_i \partial y'_k} = Q_{ik}, \qquad \frac{\partial^2 F}{\partial y'_i \partial y'_k} = R_{ik}.$$

For our further work, we introduce the following functions of s:

$$(7) \qquad \begin{aligned} \rho_{ij} &= \partial y_i(s)/\partial y'_{1j}, & \sigma_{ij} &= \partial y_i(s)/\partial y_{1j}; \\ \bar{\rho}_{ij} &= \partial \bar{y}_i(s)/\partial y'_{2j}, & \sigma_{ij} &= \partial \bar{y}_i(s)/\partial y_{2j}. \end{aligned}$$

From differentiation of both sides of equations (5) and (5*a*) with respect to the variables y_{1j} and y_{2j}, it follows that these functions satisfy the following initial conditions:

$$(8) \qquad \text{at } s = s_1 : \rho_{ij} = 0, \qquad \rho'_{ij} = \delta_{ij}; \qquad \sigma_{ij} = \delta_{ij}, \qquad \sigma'_{ij} = 0;$$

$$(8a) \qquad \text{at } s = s_2 : \bar{\rho}_{ij} = 0, \qquad \bar{\rho}'_{ij} = \delta_{ij}; \qquad \bar{\sigma}_{ij} = \delta_{ij}, \qquad \bar{\sigma}'_{ij} = 0.$$

Furthermore we differentiate both sides of equations (4) and (4*a*) with respect to the variables y_{1j} and y_{2j}, and solve the resulting equations for the derivatives of θ_h and ϕ_h; we find:

$$(9) \qquad \begin{aligned} \Delta(s_2)\frac{\partial \theta_h}{\partial y_{1j}} &= - D_{hj}(s_2), & \Delta(s_2)\frac{\partial \theta_h}{\partial y_{2j}} &= \Delta_{hj}(s_2), \\ \bar{\Delta}(s_1)\frac{\partial \phi_h}{\partial y_{1j}} &= \bar{\Delta}_{hj}(s_1), & \bar{\Delta}(s_1)\frac{\partial \phi_h}{\partial y_{2j}} &= - \bar{D}_{hj}(s_1). \end{aligned}$$

Here $\Delta(s)$ and $\bar{\Delta}(s)$ are determinants of order $n + 1$ given by the formulas

$$\Delta(s) = \begin{vmatrix} y' & \rho \\ 0 & \theta \end{vmatrix}, \qquad \bar{\Delta}(s) = \begin{vmatrix} y' & \bar{\rho} \\ 0 & \phi \end{vmatrix};$$

D_{hj} is obtained from Δ by replacing the $(h + 1)$th column by σ_{ij} $(i = 1, \cdots, n)$, 0; Δ_{hj} is the cofactor of ρ_{jh} in Δ. Similar meanings are to be attributed to $\bar{D}_{hj}$ and $\bar{\Delta}_{hj}$.

2. The second derivatives of the extremal-integral‡

It follows from the general theory that the functions ρ_{ij} and σ_{ij} form for every fixed j a solution of Jacobi's equations;§ that y' is also a solution of

* Bolza, loc. cit., p. 309.

† Loc. cit., p. 621.

‡ The deductions in this section have been materially simplified by suggestions from Professor Bliss, to whom, in particular, the simple determinantal form of the final formulas is due.

§ See, e. g., Hadamard, *Leçons sur le calcul des variations*, p. 25.

these equations has been proved by Bliss,* who has moreover shown that to every solution u of Jacobi's equations, there corresponds a unique normal solution of the form $u - \lambda y'$, characterized by the condition

$$(u - \lambda y')\,y' = 0, \qquad s_1 \leqq s \leqq s_2,$$

where λ is an uniquely determined n-partite constant.

The normal solutions which correspond to the solutions ρ_j and σ_j are given by the formulas

$$\eta_j = \rho_j - \lambda_j\,y', \qquad \zeta_j = \sigma_j - \kappa_j\,y',$$

where $\lambda_j = y'\,\rho_j$ and $\kappa_j = y'\,\sigma_j$. From equations (8) it follows that these normal solutions satisfy the following initial conditions:

$$\text{(10)} \qquad \text{at } s = s_1, \qquad \eta_{ij} = 0, \qquad \eta'_{ij} = \delta_{ij} - \theta_i\,\theta_j,$$

and

$$\text{(11) at } s = s_1, \qquad \zeta_{ij} = \delta_{ij} - \theta_i\,\theta_j, \qquad \zeta'_{ij} = -\,y''_i\,(s_1)\,\theta_j - y''_j\,(s_1)\,\theta_i.$$

From equations (10) we conclude that the functions $\eta_i\,\theta$ form for $i = 1, \cdots,$ n a normal solution whose elements vanish at $s = s_1$, together with their first derivatives; consequently they vanish identically throughout the interval $(s_1\,s_2)$; i. e., the n normal solutions η_{ij} $(j = 1, \cdots, n)$ are connected by the linear relation

$$\text{(12)} \qquad \eta_{i1}\,\theta_1 + \eta_{i2}\,\theta_2 + \cdots + \eta_{in}\,\theta_n = 0 \qquad (i = 1, \cdots, n).$$

That these normal solutions do not satisfy any other linear relation becomes evident when we notice that the coefficients c_j for such a relation would have to satisfy, in consequence of the conditions (10), the equations

$$\sum_j c_j\,(\delta_{ij} - \theta_i\,\theta_j) = 0, \qquad \text{i. e.,} \qquad c_i = \theta_i \sum_j c_j\,\theta_j.$$

Hence we would have either $\sum_j c_j\,\theta_j = 0$, from which would follow $c_i = 0$, or the fact that the c_i are proportional to the θ_i. Since there is always one non-vanishing θ_j, there is no loss in generality, if we suppose, for simplicity of notation, $\theta_n \neq 0$; in this case the solutions $\eta_1, \cdots, \eta_{n-1}$ form a matrix of $n - 1$ linearly independent solutions.

We differentiate now equations (6) and obtain

$$\Gamma_{1i,\,1j} = -\,Q_{ji}\,(s_1) - \sum_h R_{ih}\,(s_1)\,\frac{\partial\theta_h}{\partial y_{1j}},$$

$$\Gamma_{1i,\,2j} = -\sum_h R_{ih}\,(s_1)\,\frac{\partial\theta_h}{\partial y_{2j}},$$

and similar formulas for $\Gamma_{2i,\,1j}$ and $\Gamma_{2i,\,2j}$. Substituting equations (9) in

* Bliss, loc. cit., p. 201.

these formulas, we can put them into the form:

$$\Gamma_{1i,\,1j} = -\begin{vmatrix} Q_{ij} & 0 & R_i \\ \sigma_j & y' & \rho \\ 0 & 0 & \theta \end{vmatrix} \div \begin{vmatrix} y' & \rho \\ 0 & \theta \end{vmatrix}, \qquad \Gamma_{1i,\,2j} = \begin{vmatrix} y' & T_{ij} \\ 0 & \theta \end{vmatrix} \div \begin{vmatrix} y' & \rho \\ 0 & \theta \end{vmatrix},$$

where the matrix $y'\ T_{ij}$ is obtained from the matrix $y'\ \rho$ by replacing the jth row by $0, R_{i1}, \cdots, R_{in}$.

To reduce the solutions ρ_j and σ_j, occurring in these formulas, to the corresponding normal solutions, we subtract from each of the columns containing the elements of a solution ρ_j or σ_j, the column containing y' multiplied by the proper factor λ_j or κ_j; this will replace in each of the above determinants, ρ and σ by η and ζ respectively. Multiply now the last n columns in each of these modified determinants by $\theta_1, \cdots, \theta_n$ respectively, and place the sum of the results in the nth column. In view of equations (12) and of the well-known fact that $R_i\, y' = 0$, we obtain then the following

THEOREM. *The second derivatives of the extremal integral* Γ *are given by the formulas*

$$\Gamma_{1i,\,1j} = -\begin{vmatrix} Q_{ji}(s_1) & 0 & R_i(s_1) \\ \zeta_j(s_2) & y'(s_2) & H(s_2) \end{vmatrix} \div |y'(s_2)H(s_2)|,$$

$$\Gamma_{1i,\,2j} = -\sum_{h=1}^{n-1} R_{ih}(s_1)H_{jh}(s_2) \div |y'(s_2)H(s_2)|,$$

$$\Gamma_{2i,\,1j} = \sum_{h=1}^{n-1} R_{ih}(s_2)\overline{H}_{jh}(s_1) \div |y'(s_1)\overline{H}(s_1)|,$$

$$\Gamma_{2i,\,2j} = \begin{vmatrix} Q_{ji}(s_2) & 0 & R_i(s_2) \\ \overline{\zeta}_j(s_1) & y'(s_1) & \overline{H}(s_1) \end{vmatrix} \div |y'(s_1)\overline{H}(s_1)|,$$

where H *is the matrix of the* $n-1$ *linearly independent normal solutions* $\eta_1, \cdots, \eta_{n-1}$ *of Jacobi's equations characterized by conditions* (10); H_{jh} *is the cofactor of* η_{jh} *in* $|y'H|$; *where* $\overline{H}$ *and* $\overline{H}_{jh}$ *are similarly formed from the functions* $\overline{\eta}_{jh}$, *obtained by normalizing the functions* $\overline{\rho}_{jh}$; *and where* R_i *is the* $(n-1)$-*partite number* $(R_{i1}, \cdots, R_{i,\,n-1})$. *

It still remains to show that $\Gamma_{1i,\,2j} = \Gamma_{2j,\,1i}$, and that $\Gamma_{1i,\,1j}$ and $\Gamma_{2i,\,2j}$ are symmetric in i and j, i. e., we have to prove the following relations:

$$(A) \qquad -Q_{ji}(s_1)\cdot|y'H|_{s=s_2} + \sum_{h=1}^{n-1} R_{ih}(s_1)\cdot|y'Z_{hj}|_{s=s_2}$$
$$= -Q_{ij}(s_1)\cdot|y'H|_{s=s_2} + \sum_{h=1}^{n-1} R_{jh}(s_1)\cdot|y'Z_{hi}|_{s=s_2},$$

$$(B) \qquad |y'\overline{H}|_{s=s_1}\cdot\sum_{h=1}^{n-1} R_{ih}(s_1)\cdot H_{jh}(s_2) + |y'H|_{s=s_2}\cdot\sum_{h=1}^{n-1} R_{jh}(s_2)\cdot\overline{H}_{ih}(s_1),$$

* The choice of the particular matrix H of $n-1$ linearly independent normal solutions was based upon the assumption $\theta_n \neq 0$; if one of the other θ's is different from zero, an evi-

and the relation $(\bar{A})$, obtained from (A) by replacing H and Z by $\bar{H}$ and $\bar{Z}$ respectively and by permuting the arguments s_1 and s_2. Here $|y'Z_{hj}|$ is obtained from $|y'H|$ by replacing the functions η_{ih} by ζ_{ij} $(i = 1, \cdots, n)$; and $|y'\bar{Z}_{hj}|$ is similarly formed from $|y'\bar{H}|$.

To this end we begin by noticing that for any two solutions of Jacobi's equations, the relation

$$u[Qv + Rv'] - v[Qu + Ru'] = \text{constant} \tag{13}$$

holds.* Applying this formula in turn for $u = \zeta_i$, $v = \eta_j$, and for $u = \zeta_i$, $v = \zeta_j$, and making use of equations (10) and (11), we obtain the results

$$\zeta_i[Q\eta_j + R\eta_j'] - \eta_j[Q\zeta_i + R\zeta_i'] = R_{ij}(s_1),$$

and

$$\zeta_j[Q\zeta_i + R\zeta_i'] - \zeta_i[Q\zeta_j + R\zeta_j'] = Q_{ij}(s_1) - Q_{ji}(s_1),$$

by means of which formula (A) reduces to

$$|y'H| \cdot \sum_{kl} \zeta_{kj}\zeta_{li}[Q_{lk} - Q_{kl}] + \sum_{klg} \zeta_{kj}\zeta_{li}[R_{gk}H_l^g - R_{gl}H_k^g]\Big|_{s=s_2} = 0;$$

where H_k^g represents the determinant obtained from $|y'H|$ by replacing the elements of the kth row by the derivatives of the corresponding elements of the gth row. This last formula follows however immediately from equation (13) if we put $u = \xi_k$ and $v = \xi_l$, where $\xi_{ig}(s)$, $i = 1, \cdots, n$, denotes the solution of Jacobi's equations obtained by replacing the gth row of $|y'H|_{s=s_2}$ by the ith row of $|y'H|$ and dividing the result by the former determinant; we have then $H_g^l(s_2) = |y'H|_{s=s_2} \cdot \xi_{lg}'(s_2)$. Formula $(\bar{A})$ is proved in the same way, by the use of the functions $\bar{\eta}_j$, $\bar{\zeta}_j$, and $\bar{\xi}_j$ in equation (13), while formula (B) follows also from this same equation, applied to the functions ξ_j and $\bar{\xi}_i$, using the values of these functions at s_1 and at s_2.

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dent change of notation would yield another set of formulas precisely similar to those in the text. Professor Bliss has pointed out that if an arbitrarily selected set of $n - 1$ linearly independent normal solutions is used, formulas can be deduced which are more complicated than those in the text, but which apply independently of the θ which is different from zero.

* Von Escherich, Wiener Berichte, vol. 110 (1901), p. 1376; also Hadamard, loc. cit., p. 319, and Bolza, loc. cit., p. 626.

CONCERNING SINGULAR TRANSFORMATIONS B_k OF SURFACES APPLICABLE TO QUADRICS*

BY

LUIGI BIANCHI

Introduction

The theory of transformations B_k of deforms by flexure of quadrics† may be regarded as a later development of the concepts introduced by Sophus Lie in his researches on the transformations of surfaces of constant curvature. The attempt to make a possible extension of the same principles to researches of another class of applicable surfaces leads us naturally to propose a problem of arranging plane elements, or *facettes*, in space—which for the sake of clearness we will explain in detail.‡ A *facette* consists of a plane and a point of it called the *center*. We think of a surface S as the totality of its ∞^2 facettes f, the planes of the facettes being tangent to S at their respective centers. We consider associated with each facette f a *simple infinity* of facettes f', in accordance with any continuous law whatever. We imagine also that in every deformation of S, each facette f and the ∞^1 associated facettes f' are carried along as *an invariable system*. In each configuration of S the associated facettes f' form a triply infinite system, and in general they cannot be arranged into a series of ∞^1 surfaces S', each consisting of ∞^2 facettes f'. The problem in view consists precisely in determining all the cases for which the above circumstance is true *in all deformations of* S. Of the general problem thus stated it is easy to indicate an infinity of particular solutions amongst which are immediately evident those in which each of the ∞^1 surfaces S' remains constituted always of the same ∞^2 facettes f'. But, in view of the eventual applications to problems of deformation, it is opportune to limit the problem much more, and to suppose that every facette f and each of its associated facettes f' has the center of one in the plane of the other. Thus the surface S and each of its transforms S' are always the focal surfaces of the rectilinear congruence formed by the joins of corresponding points.

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† Cf. the author's *Lezioni di geometria differenziale*, vol. 3 (Pisa-Spörri, 1909).

‡ Cf. *Lezioni*, vol. 3, § 39; also my communication to the Fourth International Congress of Mathematicians held in Rome (1908) as reported in *Atti del Congresso*, vol. 2, p. 273.

Upon the problem thus limited we have had the interesting researches of A. V. Bäcklund,* which, even if they have not been useful, in essentially new cases, for the theory of applicability, they have nevertheless led to a better understanding of the reasons for the success of the method of transformations in the case of surfaces applicable to quadrics. Our aim in the present paper is an analogous one, in that by attacking a particular case of the problem, geometrically further circumscribed and definite, we arrive at complete results. The particular case which we wish to treat is that which corresponds to the special transformations B_k, indicated in my book as those *singular* transformations, which arise when, in place of quadrics homofocal to the given quadric, we use its *focal conics*. In this case, the ∞^1 facettes f' associated with a facette f of the quadric are distributed as follows: 1°. their centers are distributed on the intersection of the plane of f and the fixed plane of the focal conic; 2°. the planes of f' envelope the cone which from the center of f projects the focal conic. It is upon the first of these properties that we fix our attention, and suppose that, given a surface S_0 and a fixed plane π, with each facette f of S_0 there are associated ∞^1 facettes f', whose centers lie on the intersection of the plane of f (i. e., the tangent plane of S_0) and of π; as for the planes of f' we subject them to the single condition of passing through the center of V of f, so that they envelope a cone with vertex V, as to whose form no hypothesis is made. Under these conditions we propose to solve the following problem:

PROBLEM A. *To find what must be assumed concerning S_0 and its relation with the fixed plane π, so that, as S_0 undergoes any deformation by flexure, and each of its facettes f moves carrying with it the invariable system of ∞^1 facettes f', it will always be true that the ∞^3 facettes f' will be assembled in a series of ∞^1 surfaces S'.*

We shall demonstrate that when the case where S_0 is developable (which is of no interest) is excluded, all the solutions of problem (A) are obtained if one takes any quadric whatever which has the fixed plane π for a plane of symmetry (principal), so that one is necessarily led back to deformations of quadrics. One sees, therefore, if the quadric is general, that the distribution of the facettes is precisely the one above mentioned which presents itself in the theory of singular transformations B_k. However, when the quadric is one of revolution, *having the fixed plane for a meridian plane*, the cone enveloped by the planes of the ∞^1 facettes f' associated with a facette f breaks up into two pencils of planes. The transformed surfaces S' are in this case

* *Ueber eine Transformation von Luigi Bianchi,* Annali di Matematica, ser. 3, vol. 23 (1914); also, *Sätze aus Bianchi's Theorie der auf die Flächen zweiter Ordnung abwickelbaren Flächen, u.s.w.,* Kongl. Svenska Vetenskapsakademiens Handlingar, vol. 55, no. 22 (1916).

also applicable to one another and to the same quadric, which nevertheless is different from the given quadric, and in this respect the special case differs from the general one. The transformations which we have obtained in the special case are not any less singular transformations B_k; all of them are coördinated by the same geometrical principle, contained in problem (A).

In conclusion, the researches which we set forth in this memoir, like those already recorded by Bäcklund, have not extended the domain of transformations to a new class of applicable surfaces, but they serve to characterize all the better the rather unusual circumstance presented by the deforms of quadrics.

1. Fundamental equations of problem (A)

Let S_0 be a given surface, and x_0, y_0, z_0 the coördinates of a generic point P_0 on S_0. Assuming for curvilinear coördinates u, v on S_0 the two coördinates x_0, y_0, we write the parametric equations in the form

$$x_0 = u, \qquad y_0 = v, \qquad z_0 = z_0(u, v).$$

The first and second derivatives of z_0 with respect to u and v will be indicated by p, q, r, s, t in accordance with the notation of Monge. The linear element of S_0 is

$$ds_0^2 = (1 + p^2)\,du^2 + 2pq\,du\,dv + (1 + q^2)\,dv^2 \tag{1}$$

$$(E = 1 + p^2, \qquad F = pq, \qquad G = 1 + q^2, \qquad EG - F^2 = 1 + p^2 + q^2),$$

and the values of the Christoffel symbols for this differential form (1) are

$$\begin{aligned} \left\{ {11 \atop 1} \right\} &= \frac{pr}{1 + p^2 + q^2}, & \left\{ {12 \atop 1} \right\} &= \frac{ps}{1 + p^2 + q^2}, & \left\{ {22 \atop 1} \right\} &= \frac{pt}{1 + p^2 + q^2}, \\ \left\{ {11 \atop 2} \right\} &= \frac{qr}{1 + p^2 + q^2}, & \left\{ {12 \atop 2} \right\} &= \frac{qs}{1 + p^2 + q^2}, & \left\{ {22 \atop 2} \right\} &= \frac{qt}{1 + p^2 + q^2}. \end{aligned} \tag{2}$$

Let S be any surface applicable to S_0, with which then it has as first fundamental form (1), and S will be intrinsically defined by its second fundamental form $D\,du^2 + 2D'\,du\,dv + D''\,dv^2$. In place of these functions D, D', D'' we make use of

$$\Delta = \frac{D}{\sqrt{EG - F^2}}, \qquad \Delta' = \frac{D'}{\sqrt{EG - F^2}}, \qquad \Delta'' = \frac{D''}{\sqrt{EG - F^2}},$$

and we remark that these functions are required to satisfy only the *equation of Gauss*,

$$\Delta\Delta'' - \Delta'^2 = \frac{rt - s^2}{(1 + p^2 + q^2)^2}, \tag{3}$$

and the *equations of Codazzi*, which because of (2) are reducible to

$$(4)\qquad \begin{aligned} \frac{\partial \Delta}{\partial v} - \frac{\partial \Delta'}{\partial u} &= \frac{q}{1 + p^2 + q^2}(2s\Delta' - t\Delta - r\Delta''), \\ \frac{\partial \Delta''}{\partial u} - \frac{\partial \Delta'}{\partial v} &= \frac{p}{1 + p^2 + q^2}(2s\Delta' - t\Delta - r\Delta''). \end{aligned}$$

We retain the customary notation x, y, z for the coördinates of the moving point $P(u, v)$ on S, and X, Y, Z for the direction-cosines of the normal to S at P.

In conformity with the conditions of problem (A), we make correspond to each facette

$$f \equiv (x, y, z;\ X, Y, Z)$$

of S a simple infinity of facettes

$$f' \equiv (x', y', z';\ X', Y', Z'),$$

with centers distributed on a line in the tangent plane to S and with their planes passing through the lines joining the centers (x, y, z), (x', y', z') of the facettes. Accordingly we write the formulas

$$(5)\qquad \begin{aligned} x' &= x + l\frac{\partial x}{\partial u} + m\frac{\partial x}{\partial v}, \\ X' &= \alpha\frac{\partial x}{\partial u} + \beta\frac{\partial x}{\partial v} + \gamma X, \end{aligned}$$

with analogous expressions for the y's and z's, where $l, m; \alpha, \beta, \gamma$ are functions of u, v and *of a parameter* λ, which must remain unchanged in the deformation of S, since the ∞^1 facettes f' are rigidly attached to f.

When S takes the initial form S_0, we have

$$x = x_0 = u, \qquad y = y_0 = v, \qquad z = z_0,$$

and consequently

$$x_0' = u + l, \qquad y_0' = v + m, \qquad z_0' = z_0 + lp + mq.$$

Hence *if we assume the fixed plane π of the problem to be $x = 0$*, we must have $x_0' = 0$, thence $l = -u$, but m will remain variable. If we take as parameter $m = \lambda$, then the first equation of (5) becomes

$$(6)\qquad x' = x - u\frac{\partial x}{\partial u} + \lambda\frac{\partial x}{\partial v}.$$

We must make the further requirement that the planes of f and f' shall intersect in the join of their centers, that is

$$X'(x' - x) + Y'(y' - y) + Z'(z' - z) = 0,$$

which by means of the second equations of (5) and (6) is reducible to

$$(E\alpha + F\beta)u = (F\alpha + G\beta)\lambda.$$

If we alter α, β, γ in (5) by a factor of proportionality (to be signified hereafter by the use of the sign $\equiv$ in place of $=$), we can take

$$(7)\qquad \begin{aligned} E\alpha + F\beta &= (1 + p^2 + q^2)\lambda, \\ F\alpha + G\beta &= (1 + p^2 + q^2)u. \end{aligned}$$

From these follow

$$(7^*)\qquad \begin{aligned} \alpha &= (1 + q^2)\lambda - pqu, \\ \beta &= -pq\lambda + (1 + p^2)u, \end{aligned}$$

and then we have

$$(8)\qquad X' \equiv \alpha\frac{\partial x}{\partial u} + \beta\frac{\partial x}{\partial v} + \gamma X,$$

where α and β are functions of u, v, λ as given by (7*), and γ is an indeterminate function of u, v, λ subject to the condition that it does not vary as S is deformed.

The formulas (6) and (8) define in space ∞^3 facettes f', and we must now seek the condition to be satisfied so that, *in any deformation whatever of* S, these facettes f' can be arranged into ∞^1 surfaces S'. It is necessary and sufficient that, for every configuration S, one can determine λ as a function of u, v and *an arbitrary constant* so that the following equations are satisfied:

$$(9)\qquad \begin{aligned} X'\frac{\partial x'}{\partial u} + Y'\frac{\partial y'}{\partial u} + Z'\frac{\partial z'}{\partial u} &= 0, \\ X'\frac{\partial x'}{\partial v} + Y'\frac{\partial y'}{\partial v} + Z'\frac{\partial z'}{\partial v} &= 0. \end{aligned}$$

2. The differential equations for $\lambda = \lambda(u, v)$

The derivatives of (6) with respect to u and v lead to the expressions (cf. *Lezioni*, vol. III, p. 7)

$$(10)\qquad \begin{aligned} \frac{\partial x'}{\partial u} &= L_0\frac{\partial x}{\partial u} + M_0\frac{\partial x}{\partial v} + \frac{\partial \lambda}{\partial u}\frac{\partial x}{\partial v} + \sqrt{1 + p^2 + q^2}(\Delta'\lambda - \Delta u)X, \\ \frac{\partial x'}{\partial v} &= P_0\frac{\partial x}{\partial u} + Q_0\frac{\partial x}{\partial v} + \frac{\partial \lambda}{\partial v}\frac{\partial x}{\partial v} + \sqrt{1 + p^2 + q^2}(\Delta''\lambda - \Delta' u)X, \end{aligned}$$

where we have put

$$\begin{aligned} L_0 &= \left\{ {12 \atop 1} \right\}\lambda - \left\{ {11 \atop 1} \right\}u, & M_0 &= \left\{ {12 \atop 2} \right\}\lambda - \left\{ {11 \atop 2} \right\}u, \\ P_0 &= \left\{ {22 \atop 1} \right\}\lambda - \left\{ {12 \atop 1} \right\}u, & Q_0 &= \left\{ {22 \atop 2} \right\}\lambda - \left\{ {12 \atop 2} \right\}u + 1, \end{aligned}$$

which by (2) become

$$
\text{(11)} \qquad
\begin{aligned}
L_0 &= \frac{p(\lambda s - ur)}{1+p^2+q^2}, & M_0 &= \frac{q(\lambda s - ur)}{1+p^2+q^2},\\
P_0 &= \frac{p(\lambda t - us)}{1+p^2+q^2}, & Q_0 &= \frac{q(\lambda t - us)}{1+p^2+q^2} + 1 .
\end{aligned}
$$

In consequence of (8) and (10) the conditions (9) are reducible to

$$
\begin{aligned}
&(F\alpha + G\beta)\frac{\partial\lambda}{\partial u} + (E\alpha + F\beta)L_0 + (F\alpha + G\beta)M_0\\
&\qquad\qquad + \gamma\sqrt{1+p^2+q^2}\,(\Delta'\lambda - \Delta u) = 0,\\
&(F\alpha + G\beta)\frac{\partial\lambda}{\partial v} + (E\alpha + F\beta)P_0 + (F\alpha + G\beta)Q_0\\
&\qquad\qquad + \gamma\sqrt{1+p^2+q^2}\,(\Delta''\lambda - \Delta' u) = 0.
\end{aligned}
$$

Making use of (7) and (11), and introducing in place of $\gamma = \gamma(u, v, \lambda)$ the function

$$
\text{(12)} \qquad \Theta = \Theta(u, v, \lambda) = \frac{\gamma}{u\sqrt{1+p^2+q^2}},
$$

we give the differential equations for λ the definitive form

$$
\text{(}A\text{)} \qquad
\begin{aligned}
\frac{\partial\lambda}{\partial u} &= \frac{(qu + p\lambda)(ur - \lambda s)}{u(1+p^2+q^2)} + \Theta(\Delta u - \Delta'\lambda),\\
\frac{\partial\lambda}{\partial v} + 1 &= \frac{(qu + p\lambda)(us - \lambda t)}{u(1+p^2+q^2)} + \Theta(\Delta' u - \Delta''\lambda).
\end{aligned}
$$

We have now to examine how the unknown function $z_0 = z_0(u, v)$, which defines the initial configuration S_0 of our surface, and $\Theta = \Theta(u, v, \lambda)$, which fixes the ordering of the facettes, must be taken in order that *the equations* (A) *shall admit a solution* λ *involving an arbitrary constant, that is shall form a completely integrable system.* We remark that it must hold for any configuration of S, that is for any choice of $\Delta, \Delta', \Delta''$, provided only that the latter satisfy equations (3) and (4) of Gauss and Codazzi.

3. The conditions of integrability

If we indicate by Ω the expression obtained by taking the derivatives of the first and second equations of (A) with respect to v and u respectively and subtracting the results, we must have $\Omega = 0$, when we take account of (A). Now using the symbols $\partial/\partial u$, $\partial/\partial v$, $\partial/\partial\lambda$ for the partial derivatives with respect to u, v, λ, regarded as independent, we find

$$\Omega = \frac{\partial}{\partial u}\left[\frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)}\right] + \frac{\partial}{\partial v}\left[\frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)}\right]$$
$$+ \frac{2pt\lambda + u(qt - ps)}{u(1 + p^2 + q^2)}\frac{\partial\lambda}{\partial u} + \frac{-2ps\lambda + u(pr - qs)}{u(1 + p^2 + q^2)}\frac{\partial\lambda}{\partial v}$$
$$+ \frac{\partial\Theta}{\partial u}(\Delta''\lambda - \Delta' u) + \frac{\partial\Theta}{\partial v}(\Delta u - \Delta'\lambda) + \frac{\partial\Theta}{\partial\lambda}\left\{(\Delta''\lambda - \Delta' u)\frac{\partial\lambda}{\partial u}\right.$$
$$\left. + (\Delta u - \Delta'\lambda)\frac{\partial\lambda}{\partial v}\right\} + \Theta u\left(\frac{\partial\Delta}{\partial v} - \frac{\partial\Delta'}{\partial u}\right) + \Theta\lambda\left(\frac{\partial\Delta''}{\partial u} - \frac{\partial\Delta'}{\partial v}\right)$$
$$+ \Theta\Delta''\frac{\partial\lambda}{\partial u} - \Theta\Delta'\left(\frac{\partial\lambda}{\partial v} + 1\right).$$

If we substitute for the differences $(\partial\Delta/\partial v) - (\partial\Delta'/\partial u)$, $(\partial\Delta''/\partial u) - (\partial\Delta'/\partial v)$ their values as given by the equations (4) of Codazzi, and for $\partial\lambda/\partial u$, $\partial\lambda/\partial v$ those which follow from (A), we have

$$\Omega = \frac{\partial}{\partial u}\left[\frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)}\right] + \frac{\partial}{\partial v}\left[\frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)}\right]$$
$$+ \frac{2pt\lambda + u(qt - ps)}{u(1 + p^2 + q^2)}\left\{\frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} + \Theta(\Delta u - \Delta'\lambda)\right\}$$
$$+ \frac{-2ps\lambda + u(pr - qs)}{u(1 + p^2 + q^2)}\left\{-1 + \frac{(qu + p\lambda)(us - \lambda t)}{u(1 + p^2 + q^2)}\right.$$
$$\left. + \Theta(\Delta' u - \Delta''\lambda)\right\} + \frac{\partial\Theta}{\partial u}(\Delta''\lambda - \Delta' u) + \frac{\partial\Theta}{\partial v}(\Delta u - \Delta'\lambda) \tag{13}$$
$$+ \frac{\partial\Theta}{\partial\lambda}\left\{(\Delta u - \Delta'\lambda)\left(-1 + \frac{(qu + p\lambda)(us - \lambda t)}{u(1 + p^2 + q^2)}\right)\right.$$
$$\left. + \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)}(\Delta''\lambda - \Delta' u)\right\}$$
$$+ \frac{\Theta(qu + p\lambda)}{1 + p^2 + q^2}\left(2s\Delta' - t\Delta - r\Delta'' + \frac{ur - \lambda s}{u}\Delta'' + \frac{\lambda t - us}{u}\Delta'\right)$$
$$+ \Theta^2[(\Delta u - \Delta'\lambda)\Delta'' - (\Delta' u - \Delta''\lambda)\Delta'].$$

Because of the equation (4) of Gauss the coefficient of Θ^2 in equation (13) is reducible to $u(rt - s^2)/(1 + p^2 + q^2)^2$, and Ω is seen to be linear in Δ, Δ', Δ'', so that we may write

$$\Omega = a\Delta + b\Delta' + c\Delta'' + d, \tag{13*}$$

where a, b, c, d are functions of u, v, λ. Thus on our hypothesis that $\Omega = 0$

whatever be $\Delta, \Delta', \Delta''$, provided only that they satisfy the Gauss and Codazzi equations, it follows from known considerations (cf. *Lezioni*, vol. 2, p. 254) that the functions a, b, c, d must vanish separately, and that whatever be λ. We turn to the discussion of these conditions.

4. The function Θ^2 as a polynomial of the second degree in λ

We commence with the fourth coefficient d, which from the above observations is found to be

$$d = \Theta^2 \frac{u(rt - s^2)}{(1 + p^2 + q^2)^2} + \frac{\partial}{\partial u}\left[\frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)}\right] + \frac{\partial}{\partial v}\left[\frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)}\right] + \frac{2ps\lambda + u(qs - pr)}{u(1 + p^2 + q^2)}$$
$$+ \frac{qu + p\lambda}{u^2(1 + p^2 + q^2)^2}\{(ur - \lambda s)[2pt\lambda + u(qt - ps)] + (us - \lambda t)[-2ps\lambda + u(pr - qs)]\}.$$

In the last term the coefficient within the brackets $\{\ \}$ of

$$\frac{qu + p\lambda}{u^2(1 + p^2 + q^2)^2}$$

is reducible to an expression linear in λ, namely

$$u(qu + p\lambda)(rt - s^2),$$

and, if the differentiation indicated is carried out, it is found that the third derivatives of z_0 cancel one another and the resulting expression for d is

$$d = \Theta^2 u \frac{rt - s^2}{(1 + p^2 + q^2)^2} + \left\{\frac{(rt - s^2)(1 - p^2 + q^2)}{u(1 + p^2 + q^2)^2} - \frac{pt}{u^2(1 + p^2 + q^2)}\right\}\lambda^2$$
$$- \frac{4pq(rt - s^2)}{(1 + p^2 + q^2)^2}\lambda + \frac{u(rt - s^2)(1 + p^2 - q^2)}{(1 + p^2 + q^2)^2} + \frac{2ps}{u(1 + p^2 + q^2)}\lambda$$
$$- \frac{pr}{1 + p^2 + q^2} + \frac{rt - s^2}{u(1 + p^2 + q^2)^2}(p^2\lambda^2 + 2pqu\lambda + q^2u^2).$$

On the assumption that S_0 is not a developable surface we have $rt - s^2 \neq 0$, so that when d is equated to zero, we find that Θ^2 is a quadratic in λ of the form

$$\Theta^2 = A\lambda^2 + 2B\lambda + C, \tag{14}$$

where the functions A, B, C have the values

$$
\begin{aligned}
A &= \frac{pt(1+p^2+q^2)}{u^3(rt-s^2)} - \frac{1+q^2}{u^2}, \\
B &= -\frac{ps(1+p^2+q^2)}{u^2(rt-s^2)} + \frac{pq}{u}, \qquad (15) \\
C &= \frac{pr(1+p^2+q^2)}{u(rt-s^2)} - (1+p^2).
\end{aligned}
$$

Because of the form (14) of Θ^2 one has at once a consequence with respect to the type of cone enveloped by the planes of the ∞^1 facettes f' associated with a facette f. The formula (8), which gives the direction-cosines of the planes of f' is reducible by (7*), (12), and (14) to

$$
\begin{aligned}
X' \equiv \{(1+q^2)\lambda - pqu\}\frac{\partial x}{\partial u} &+ \{-pq\lambda + (1+p^2)u\}\frac{\partial x}{\partial v} \\
&+ u\sqrt{1+p^2+q^2}\sqrt{A\lambda^2+2B\lambda+C}\cdot X.
\end{aligned} \tag{16}
$$

Since, for fixed values of u and v, these expressions are linear in λ and the radical $\sqrt{A\lambda^2+2B\lambda+C}$, we have the result:

The cone enveloped by the planes of the ∞^1 facettes f', associated with a facette f, must be a quadric cone (with vertex at the center of f).

5. The differential equations for A, B, C

We calculate now the other three coefficients a, b, c in (13*) which also must be equated to zero. It is found that

$$
\begin{aligned}
a &= u\frac{\partial\Theta}{\partial v} + \left\{\frac{(qu+p\lambda)(us-\lambda t)}{1+p^2+q^2} - u\right\}\frac{\partial\Theta}{\partial\lambda} + \frac{p(\lambda t - us)}{1+p^2+q^2}\Theta, \\
b &= -u\frac{\partial\Theta}{\partial u} - \lambda\frac{\partial\Theta}{\partial v} + \left\{\frac{(qu+p\lambda)(\lambda t-us)}{u(1+p^2+q^2)}\lambda + \lambda\right. \\
&\quad \left. + \frac{(qu+p\lambda)(\lambda s-ur)}{1+p^2+q^2}\right\}\frac{\partial\Theta}{\partial\lambda} + \frac{2(qu+p\lambda)}{1+p^2+q^2}\Theta \\
&\quad + \frac{(qu+p\lambda)(\lambda t-us)}{u(1+p^2+q^2)}\Theta + \frac{-2p\lambda s + u(pr-qs)}{1+p^2+q^2}\Theta \\
&\quad - \lambda\frac{2pt\lambda + u(qt-ps)}{u(1+p^2+q^2)}\Theta, \\
c &= \lambda\left\{\frac{\partial\Theta}{\partial u} + \frac{(qu+p\lambda)(ur-\lambda s)}{u(1+p^2+q^2)}\frac{\partial\Theta}{\partial\lambda} + \frac{p(\lambda s-ur)}{u(1+p^2+q^2)}\Theta\right\}.
\end{aligned}
$$

It is readily seen that b is a linear combination of a and c, so that there remains for consideration only $a=0$, $c=0$. If these be multiplied by Θ,

we have on account of (14)

$$\Theta\frac{\partial\Theta}{\partial\lambda} = A\lambda + B,$$

$$\Theta\frac{\partial\Theta}{\partial u} = \frac{1}{2}\frac{\partial A}{\partial u}\lambda^2 + \frac{\partial B}{\partial u}\lambda + \frac{1}{2}\frac{\partial C}{\partial u},$$

$$\Theta\frac{\partial\Theta}{\partial v} = \frac{1}{2}\frac{\partial A}{\partial v}\lambda^2 + \frac{\partial B}{\partial v}\lambda + \frac{1}{2}\frac{\partial C}{\partial v};$$

so that these two equations are equivalent to

$$u\left(\frac{1}{2}\frac{\partial A}{\partial v}\lambda^2 + \frac{\partial B}{\partial v}\lambda + \frac{1}{2}\frac{\partial C}{\partial v}\right) + (A\lambda + B)\left(\frac{(qu + p\lambda)(us - \lambda t)}{1 + p^2 + q^2} - u\right)$$
$$+ \frac{p(\lambda t - us)}{1 + p^2 + q^2}(A\lambda^2 + 2B\lambda + C) = 0,$$

$$\frac{1}{2}\frac{\partial A}{\partial u}\lambda^2 + \frac{\partial B}{\partial u}\lambda + \frac{1}{2}\frac{\partial C}{\partial u} + (A\lambda + B)\frac{(qu + p\lambda)(ur - \lambda s)}{1 + p^2 + q^2}$$
$$+ \frac{p(\lambda s - ur)}{u(1 + p^2 + q^2)}(A\lambda^2 + 2B\lambda + C) = 0.$$

The first members of these equations (since the terms in λ^3 cancel one another) are quadratics in λ, which from our hypothesis must *vanish identically*. It results that *all* the first derivatives of A, B, and C are expressible linearly in terms of A, B, C as follows:

$$(17)\qquad \begin{aligned}
\frac{\partial A}{\partial u} &= \frac{2qs}{1 + p^2 + q^2}A - \frac{2ps}{u(1 + p^2 + q^2)}B,\\
\frac{\partial B}{\partial u} &= \frac{-qur}{1 + p^2 + q^2}A + \frac{pr + qs}{1 + p^2 + q^2}B - \frac{ps}{u(1 + p^2 + q^2)}C,\\
\frac{\partial C}{\partial u} &= \frac{-2qur}{1 + p^2 + q^2}B + \frac{2pr}{1 + p^2 + q^2}C,\\
\frac{\partial A}{\partial v} &= \frac{2qt}{1 + p^2 + q^2}A - \frac{2pt}{u(1 + p^2 + q^2)}B,\\
\frac{\partial B}{\partial v} &= \left(1 - \frac{qus}{1 + p^2 + q^2}\right)A + \frac{ps + qt}{1 + p^2 + q^2}B - \frac{pt}{u(1 + p^2 + q^2)}C,\\
\frac{\partial C}{\partial v} &= 2\left(1 - \frac{qus}{1 + p^2 + q^2}\right)B + \frac{2ps}{1 + p^2 + q^2}C.
\end{aligned}$$

Now, by means of (15), the functions A, B, C are expressible in terms of a single unknown function z_0 and its first and second derivatives p, q, r, s, t, so that the system (17) is a system of the *third order in partial derivatives* of z_0, which must be studied and integrated. But before doing this it will be useful to obtain from these equations a consequence with regard to the discriminant $\nabla = B^2 - AC$ of the quadratic Θ^2 in λ. With the aid of (17) we find

$$\frac{\partial \nabla}{\partial u} = \frac{2(pr + qs)}{1 + p^2 + q^2}\nabla = \nabla \cdot \frac{\partial}{\partial u}\log(1 + p^2 + q^2),$$

$$\frac{\partial \nabla}{\partial v} = \frac{2(ps + qt)}{1 + p^2 + q^2}\nabla = \nabla \cdot \frac{\partial}{\partial v}\log(1 + p^2 + q^2),$$

whence on integration

$$\nabla = B^2 - AC = k(1 + p^2 + q^2), \tag{18}$$

where *k is a constant.* In the geometrical problem there is an essential difference between the cases $k \neq 0$, and $k = 0$. In the first case ($k \neq 0$), as Θ^2 is not a perfect square, the planes of the ∞^1 facettes f', associated with a facette f, envelope a quadric cone. In the second case ($k = 0$), the polynomial Θ^2 is a perfect square and the cone decomposes into two pencils of planes.

6. Application of the transformation of Legendre

For the integration of the system (17), where A, B, C have the values (15), it is opportune to effect a change of the independent variables and of the unknown function given by the *transformation of Legendre,* as suggested by the form of the system (17). Thus we take p and q for independent variables (which is possible as S_0 is not developable), and for the unknown function

$$Z_0 = pu + qv - z_0.$$

The notation of Monge, namely P, Q; R, S, T, referring to the first and second derivatives of Z_0 with respect to p and q, will be used as above, and we have the known formulas

$$P = u, \qquad Q = v, \qquad R = \frac{t}{rt - s^2}, \qquad S = \frac{-s}{rt - s^2}, \qquad T = \frac{r}{rt - s^2}.$$

Now

$$Rr + Ss = 1, \qquad Sr + Ts = 0,$$

$$Rs + St = 0, \qquad Ss + Tt = 1,$$

so that the formulas (15) become

$$
(19) \qquad \begin{aligned}
A &= \frac{p(1+p^2+q^2)}{P^3} R - \frac{1+q^2}{P^2}, \\
B &= \frac{p(1+p^2+q^2)}{P^2} S + \frac{pq}{P}, \\
C &= \frac{p(1+p^2+q^2)}{P} T - (1+p^2).
\end{aligned}
$$

By transforming equations (17) in accordance with the formulas of differentiation

$$
\frac{\partial}{\partial p} = R\frac{\partial}{\partial u} + S\frac{\partial}{\partial v}, \qquad \frac{\partial}{\partial q} = S\frac{\partial}{\partial u} + T\frac{\partial}{\partial v},
$$

we have the result

$$
\begin{aligned}
\frac{\partial A}{\partial p} &= 0, \\
\frac{\partial A}{\partial q} &= \frac{2q}{1+p^2+q^2} A - \frac{2p}{P(1+p^2+q^2)} B, \\
\frac{\partial B}{\partial p} &= \left(S - \frac{qP}{1+p^2+q^2}\right) A + \frac{p}{1+p^2+q^2} B, \\
\frac{\partial B}{\partial q} &= TA + \frac{q}{1+p^2+q^2} B - \frac{p}{P(1+p^2+q^2)} C, \\
\frac{\partial C}{\partial p} &= 2\left(S - \frac{qP}{1+p^2+q^2}\right) B + \frac{2p}{1+p^2+q^2} C, \\
\frac{\partial C}{\partial q} &= 2TB.
\end{aligned}
$$

Finally, if the expressions from (19) are substituted in these equations, we find that the six first derivatives of R, S, T, necessarily satisfying the conditions

$$
\frac{\partial R}{\partial q} = \frac{\partial S}{\partial p}, \qquad \frac{\partial S}{\partial q} = \frac{\partial T}{\partial p},
$$

are given by the following equations:

$$
(B) \qquad \begin{aligned}
\frac{\partial R}{\partial p} &= \frac{3}{P} R^2 - \frac{3R}{p}, \\
\frac{\partial R}{\partial q} &= \frac{\partial S}{\partial p} = 3\frac{RS}{P} - 2\frac{S}{p}, \\
\frac{\partial S}{\partial q} &= \frac{\partial T}{\partial p} = 2\frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p}, \\
\frac{\partial T}{\partial q} &= 3\frac{ST}{P}.
\end{aligned}
$$

This differential system (B) gives all the *third* derivatives of Z_0 expressed in terms of the first and second derivatives, P, Q, R, S, T. Moreover, the conditions of integrability of equations (B), namely

$$\frac{\partial}{\partial q}\left(\frac{3}{P}R^2 - \frac{3R}{p}\right) = \frac{\partial}{\partial p}\left(3\frac{RS}{P} - 2\frac{S}{p}\right),$$

$$\frac{\partial}{\partial q}\left(3\frac{RS}{P} - 2\frac{S}{p}\right) = \frac{\partial}{\partial p}\left(2\frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p}\right),$$

$$\frac{\partial}{\partial q}\left(2\frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p}\right) = \frac{\partial}{\partial p}\left(3\frac{ST}{P}\right),$$

are readily found to be satisfied in consequence of (B). Hence the system (B) is completely integrable, so that from this point of view we know that the problem (A) admits *solutions depending on six arbitrary constants,* since, in fact, the values of the unknown function and its five first and second derivatives are arbitrary for initial values of the independent variables.

7. Integration of the differential system (B)

In order to integrate the system (B) the ordinary method of solution of a completely integrable system could be used, but a more expeditious course is the following. From (B) we take the three equations

$$\frac{\partial R}{\partial p} = \frac{3R^2}{P} - \frac{3R}{p}, \qquad \frac{\partial S}{\partial p} = 3\frac{RS}{P} - \frac{2S}{p}, \qquad \frac{\partial T}{\partial q} = 3\frac{ST}{P},$$

which give by a first integration

$$R = \frac{P^3}{p^3}\psi(q), \qquad S = \frac{P^3}{p^2}\theta(q), \qquad T = P^3\phi(p), \tag{20}$$

when $\psi(q)$ and $\theta(q)$ are functions of q alone and $\phi(p)$ is a function of p alone. These functions must be such that the system (B) is satisfied. For this we observe in the first place that since $dP = Rdp + Sdq$, we have from (20),

$$\frac{dP}{P^3} = \frac{\psi(q)\,dp}{p^3} + \frac{\theta(q)\,dq}{p^2}. \tag{21}$$

The condition of integrability of the second member necessitates

$$\theta(q) = -\tfrac{1}{2}\psi'(q), \qquad \left(\psi'(q) \equiv \frac{d\psi}{dq}\right),$$

so that (21) can be integrated in the form

$$\frac{1}{P^2} = \frac{\psi(q)}{p^2} + c,$$

c being constant, or

$$\frac{p^2}{P^2} = \psi(q) + cp^2. \tag{22}$$

Thus we have

$$S = -\frac{1}{2}\frac{P^3}{p^2}\psi'(q),$$

whence

$$\frac{\partial S}{\partial q} = \frac{3}{4}\frac{P^5}{p^4}\psi'^2(q) - \frac{1}{2}\frac{P^3}{p^2}\psi''(q),$$

so that the third of (B) becomes

$$\frac{1}{4}\frac{P^5}{p^4}\psi'^2(q) - \frac{1}{2}\frac{P^3}{p^2}\psi''(q) = \frac{P^5}{p^3}\phi(p)\psi(q) - \frac{P^3}{p}\phi(p).$$

This equation multiplied by $4p^4/P^5$ is reducible by means of (22) to

$$2\psi''(q)(\psi(q) + cp^2) - \psi'^2(q) = 4cp^3\phi(p). \tag{23}$$

We shall find that the cases $c \neq 0$ and $c = 0$ have different geometrical significance, and so we treat these cases separately.

1° *Case.* $c \neq 0$. If in (23) we take for q any constant value whatever, then necessarily

$$p^3\phi(p) = \alpha p^2 + \beta,$$

where α and β are constants, and from (23) results $\psi''(q) = 2\alpha$, so that

$$\psi(q) = \alpha q^2 + 2\gamma q + \delta, \tag{24}$$

where γ and δ are new constants. In order that these values shall satisfy (23) it is necessary and sufficient that the constants $c, \alpha, \beta, \gamma, \delta$ be in the relation

$$\alpha\delta - \gamma^2 = c\beta, \tag{25}$$

whence we have the definitive formulas

$$\begin{gathered} R = \frac{P^3}{p^3}(\alpha q^2 + 2\gamma q + \delta), \qquad S = -\frac{P^3}{p^2}(\alpha q + \gamma), \\ T = \frac{P^3}{p^3}(\alpha p^2 + \beta), \end{gathered} \tag{26}$$

together with

$$\frac{p^2}{P^2} = \alpha q^2 + 2\gamma q + \delta + cp^2. \tag{27}$$

Now we observe that conversely these expressions, in which the constants are in the relation (25), satisfy all of equations (B), including the last which was not considered above.

2° *Case.* $c = 0$. We will show that the expressions (26) hold also in this case when in them we put $c = 0$. In fact, (23) reduces now to

$$2\psi(q)\cdot\psi''(q) = \psi'^2(q),$$

of which the integral is given by (24) with $\alpha\delta - \gamma^2 = 0$, in conformity with (25). At the same time we find that the first two of (26) hold also. With respect to the third we take $T = P^3\phi(p)$ as given by (20), subject to the condition

$$\frac{\partial T}{\partial p} = 2\frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p},$$

for the determination of $\phi(p)$. This gives

$$\frac{2P^5}{p^3}(\alpha q^2 + 2\gamma q + \delta)\phi(p) + P^3\left(\phi'(p) - \frac{\phi(p)}{p}\right)$$
$$= 2\frac{P^5}{p^2}(\alpha q + \gamma)^2 - \frac{P^3}{p}\phi(p).$$

Multiplying by p^4/P^5 and reducing with the aid of (27), we get

$$p^3\phi'(p) + 3p^2\phi(p) = 2\alpha p,$$

of which the integral is

$$p^3\phi(p) = \alpha p^2 + \beta,$$

where β is a constant. This gives the expression (26) for T, and consequently (26) and (27) hold when $c = 0$.

8. Geometric interpretation of the solution

It remains for us to complete the integration by calculating by quadratures Q and Z_0 from (26) and (27), which process introduces two new arbitrary constants. Thus we have the complete solution of the problem (A) which depends effectively on six arbitrary constants, as remarked in § 6. We have now to give the geometric interpretation of these results. In doing so, we separate the cases $c \neq 0$ and $c = 0$.

1° *Case.* $c \neq 0$. Substituting from (26) and (27) in $dQ = Sdp + Tdq$, we have

$$dQ = \frac{-p(\alpha q + \gamma)dp + (\alpha p^2 + \beta)dq}{(\alpha q^2 + 2\gamma q + \delta + cp^2)^{3/2}},$$

of which the integral is

$$Q = \frac{\alpha q + \gamma}{c(\alpha q^2 + 2\gamma q + \delta + cp^2)^{1/2}} + h,$$

h being a constant.

In like manner

$$dZ_0 = Pdp + Qdq = \frac{cpdp + (\alpha q + \gamma)dq}{c(\alpha q^2 + 2\gamma q + \delta + cp^2)^{1/2}} + hdq,$$

whence we have

$$Z_0 = \frac{1}{c}\sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2} + hq + k,$$

where k is a constant.

Having regard to the formulas of transformation of Legendre (§ 6), we can write the parametric equations of our surface S_0, expressing the coördinates x_0, y_0, z_0 of a moving point in terms of parameters p and q, in the form

$$(28) \qquad x_0 = P, \qquad y_0 = Q, \qquad z_0 = pP + qQ - Z_0,$$

which in consequence of the preceding are reducible to

$$x_0 = \frac{p}{\sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2}}, \qquad y_0 = \frac{\alpha q + \gamma}{c\sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2}} + h,$$

$$z_0 + k = \frac{-(\gamma q + \delta)}{c\sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2}}.$$

These equations, because of the relation (25) between the constants, define the *central quadric* with the equation

$$(29) \quad \beta x_0^2 + \delta(y_0 - h)^2 + 2\gamma(y_0 - h)(z_0 + k) + \alpha(z_0 + k)^2 = \frac{\beta}{c},$$

for which the plane $x = 0$ is a principal plane. Conversely, we note that the presence of six arbitrary constants enables us to identify the quadric (29) with any central quadric *whatever* for which $x = 0$ is a plane of symmetry. It should be observed that we must take $\beta \neq 0$, otherwise the quadric degenerates into a pair of planes.

2° *Case.* $c = 0$. Since $\alpha\delta - \gamma^2 = 0$, the expression $\alpha q^2 + 2\gamma q + \delta$ is a perfect square of a linear expression, say $aq + b$ (a and b constants), with $\alpha = a^2$, $\gamma = ab$, $\delta = b^2$, and the formulas for P, R, S, T become

$$(30) \quad P = \frac{p}{aq + b}, \qquad R = \frac{1}{aq + b}, \qquad S = \frac{-ap}{(aq + b)^2}, \qquad T = \frac{a^2p^2 + \beta}{(aq + b)^3}.$$

In order to obtain the expressions for Q and Z_0, it is necessary to distinguish between the cases $a \neq 0$ and $a = 0$. If $a \neq 0$, we find successively

$$Q = \frac{-(a^2p^2 + \beta)}{2a(aq + b)^2} + h, \qquad Z_0 = \frac{a^2p^2 + \beta}{2a^2(aq + b)} + hq + k,$$

where h and k are constants. Then S_0 is given by

$$x_0 = \frac{p}{aq + b}, \qquad y_0 = h - \frac{a^2p^2 + \beta}{2a(aq + b)^2}, \qquad z_0 = \frac{a^2bp^2 - \beta(2aq + b)}{2a^2(aq + b)^2} - k.$$

Eliminating p and q, we find that S_0 is the *paraboloid*

$$a\{a(z_0 + k) + b(y_0 - h)\}^2 + \beta\{2(y_0 - h) + ax_0^2\} = 0,$$

for which $x = 0$ is the plane of symmetry, and furthermore it can be made to coincide with any paraboloid possessing this property.

Analogous results follow in the case $a = 0$. Then we have

$$P = \frac{p}{b}, \qquad R = \frac{1}{b}, \qquad S = 0, \qquad T = \frac{\beta}{b^3},$$

whence

$$Q = \frac{\beta q}{b^3} + h, \qquad Z_0 = \frac{p^2}{2b} + \frac{\beta q^2}{2b^3} + hq + k \quad (h, k \text{ constants}).$$

It results that

$$x_0 = \frac{p}{b}, \qquad y_0 = \frac{\beta q}{b^3} + h, \qquad z_0 = \frac{p^2}{2b} + \frac{\beta q^2}{2b^3} - k,$$

from which it is seen that S_0 is the paraboloid

$$2(z_0 + k) = bx_0^2 + \frac{b^3}{\beta}(y_0 - h)^2.$$

Hence our researches as to the character of S_0 which solves problem (*A*) yield the final result:

The surfaces which solve problem A are all the quadrics having the fixed plane for plane of symmetry and only these.

It remains for us to examine, for each type of quadric, how the ∞^1 facettes f', associated with a facette f, must be distributed. This distribution, as results from formulas (15), § 4, is fully determined; it is quadratic. If we wish to utilize properties already known in the theory of singular transformations B_k for the deforms of quadrics, we can at once infer, *at least for the general quadrics*, how the facettes f' must be distributed. But it is our aim, on the contrary, to discover the law of distribution of the facettes as a consequence of the data of the problem. Thus also in a new way we shall establish the foundations of the theory of (singular) transformations B_k.

9. Distribution of facettes in the case of the general paraboloid

We commence with the case of the paraboloid whose parametric equations can be taken in the normal form

$$x_0 = u, \qquad y_0 = v, \qquad z_0 = \tfrac{1}{2}(au^2 + bv^2) \quad (a, b \text{ constants}).$$

Now we have

$$p = au, \qquad q = bv, \qquad r = a, \qquad s = 0, \qquad t = b, \qquad rt - s^2 = ab,$$

and formulas (15) are simply

$$(31) \qquad A = a^2, \qquad B = abv, \qquad C = \frac{a-b}{b}(1 + a^2 u^2) + abv^2.$$

It results that

$$\nabla = B^2 - AC = \left(1 - \frac{a}{b}\right)(1 + a^2 u^2 + b^2 v^2) = \left(1 - \frac{a}{b}\right)(1 + p^2 + q^2),$$

which is in conformity with the general formula (18) § 5, with $k = 1 - (a/b)$, so that $k = 0$ only in the case of the paraboloid of rotation, excluded for the present and considered in particular in § 11.

Hence in the case of the general paraboloid $(a \neq b)$, the polynomial

$$(32) \qquad \Theta^2 = a^2\lambda^2 + 2abv\lambda + \frac{a-b}{b}(1 + a^2 u^2) + abv^2$$

is not a perfect square, so that the planes of the ∞^1 facettes associated with a facette f actually envelop a quadric cone (§ 5), whose geometric character is to be determined. It is necessary for this to recur to formulas (16), § 4, which assign values to X', Y', Z', proportional to the direction-cosines of the normal to the plane of a facette f', and since the form of the cone is independent of the deformation of S, we can consider S in the initial configuration S_0. Thus we have

$$x = u, \qquad y = v, \qquad z = \tfrac{1}{2}(au^2 + bv^2), \qquad p = au, \qquad q = bv,$$

$$X = \frac{-p}{\sqrt{1 + p^2 + q^2}}, \qquad Y = \frac{-q}{\sqrt{1 + p^2 + q^2}}, \qquad Z = \frac{1}{\sqrt{1 + p^2 + q^2}},$$

so that (16) become

$$(33) \qquad \begin{aligned} X' &\equiv (1 + b^2 v^2)\lambda - abu^2 v - au^2\Theta, \\ Y' &\equiv -abuv\lambda + (1 + a^2 u^2)u - buv\Theta, \\ Z' &\equiv au\lambda + buv + u\Theta, \end{aligned}$$

where

$$(33^*) \qquad \Theta = \sqrt{a^2\lambda^2 + 2abv\lambda + \frac{a-b}{b}(1 + a^2 u^2) + abv^2}.$$

The equation of the plane enveloping the cone may be written, x, y, z indicating current coördinates,

$$(x - u)X' + (y - v)Y' + (z - \tfrac{1}{2}(au^2 + bv^2))Z' = 0.$$

This cuts the plane $x = 0$ in the line

$$(34) \qquad Y'y + Z'z = uX' + vY' + \tfrac{1}{2}(au^2 + bv^2)Z',$$

and the section of the cone made by the plane $x = 0$ will be the conic enveloped by the line (34), as it moves with the variation of λ. We shall find

that this conic remains the same for all values of u and v, and coincides with the *focal parabola* in the plane $x = 0$. The equation in point coördinates of this parabola is

$$y^2 = \left(\frac{1}{b} - \frac{1}{a}\right)\left(2z - \frac{1}{a}\right),$$

and therefore its equation in homogeneous line coördinates ξ, η, ζ is

$$(b - a)\xi^2 + b\eta^2 + 2ab\eta\zeta = 0. \tag{35}$$

From the expressions (33) for X', Y', Z' and (33*) for Θ it is seen that *for all values of u, v, λ* the identity

$$(b - a)Y'^2 + bZ'^2 - 2abZ'(uX' + vY' + \tfrac{1}{2}(au^2 + bv^2)Z') = 0,$$

holds, and therefore the line (34) with coördinates

$$\xi = Y', \qquad \eta = Z', \qquad \zeta = -(uX' + vY' + \tfrac{1}{2}(au^2 + bv^2)Z')$$

envelops the focal parabola (35). Consequently we have the result:

In the case of the general paraboloids, the planes of the facettes f', associated with a facette f of the paraboloid, envelop the cone which from the center of f projects the focal parabola in the fixed plane containing the centers of the facettes f'.

10. Case of the general central quadric

We come now to the other case where S_0 is a central quadric with the fixed plane $x = 0$ for a principal plane. Its parametric equations can be written in the normal form

$$x_0 = u, \qquad y_0 = v, \qquad z_0 = \sqrt{au^2 + bv^2 + c} \quad (a, b, c \text{ constants}),$$

whence we have

$$p = \frac{au}{z_0}, \qquad q = \frac{bv}{z_0}, \qquad 1 + p^2 + q^2 = \frac{z_0^2 + a^2u^2 + b^2v^2}{z_0^2},$$

$$r = \frac{a(bv^2 + c)}{z_0^3}, \qquad s = -\frac{abuv}{z_0^3}, \qquad t = \frac{b(au^2 + c)}{z_0^3}, \qquad rt - s^2 = \frac{abc}{z_0^4}.$$

From the general formulas (15), § 4, we deduce at once the following expressions for A, B, C:

$$\begin{aligned} A &= a\frac{a(a+1)u^2 + b(b+1)v^2 + c(a+1)}{cz_0^2}, \\ B &= av\frac{a(a+1)u^2 + b(b+1)v^2 + c(b+1)}{cz_0^2}, \\ C &= \frac{a(bv^2 + c)[a(a+1)u^2 + b(b+1)v^2 + c] - bc[a(a+1)u^2 + bv^2 + c]}{bcz_0^2}, \end{aligned} \tag{36}$$

and the discriminant $\nabla = B^2 - AC$ of the polynomial

$$\Theta^2 = A\lambda^2 + 2B\lambda + C$$

is easily found to be

$$B^2 - AC = k\frac{z_0^2 + a^2 u^2 + b^2 v^2}{z_0^2} = k(1 + p^2 + q^2),$$

where k has the value

$$k = \frac{a}{bc}(a + 1)(a - b). \tag{37}$$

This result is in conformity with the general formula (18), § 5, and one sees that $k = 0$ only in case $a = -1$ or $a = b$, in which cases S_0 is a quadric of revolution with the fixed plane $x = 0$ for a meridian plane. Hence when the quadric is general, *in the sense that it is not a quadric of revolution about an axis contained in the fixed plane,* the planes of the ∞^1 facettes f', associated with a facette f, envelop a quadric cone (§ 5). We have yet to show that *this cone is the one which projects from the center of f the focal conic in the plane $x = 0$.*

Proceeding as in the case of the paraboloids we calculate from (16) the values of X', Y', Z' when the surface S takes the initial configuration of the quadric S_0. Multiplying the values of X', Y', Z' by the factor z_0^2, we find

$$X' \equiv (z_0^2 + b^2 v^2)\lambda - abu^2 v - au^2 z_0 \Theta, \qquad \Theta = \sqrt{A\lambda^2 + 2B\lambda + C},$$

$$Y' \equiv -abuv\lambda + (z_0^2 + a^2 u^2)u - buvz_0\Theta, \tag{38}$$

$$Z' \equiv auz_0\lambda + buvz_0 + uz_0^2\Theta.$$

The plane of the facette f' intersects the plane $x = 0$ in the line whose coördinates are

$$\xi = Y', \qquad \eta = Z', \qquad \zeta = -(uX' + vY' + z_0 Z').$$

The equation in point coördinates, of the focal conic in the plane $x = 0$ is

$$\frac{aby^2}{c(b - a)} + \frac{az^2}{c(a + 1)} = 1,$$

and in line coördinates

$$c(b - a)\xi^2 + bc(a + 1)\eta^2 - ab\zeta^2 = 0.$$

It is easily verified that the expressions (38) satisfy identically the equation

$$c(b - a)Y'^2 + bc(a + 1)Z'^2 - ab(uX' + vY' + z_0 Z')^2 = 0,$$

so that the cone enveloped by the planes of the facettes f', associated with f with center u, v, z_0, is the one which from this plane projects the focal conic.

11. Case of quadrics of revolution

We consider finally the case previously excluded where S_0 is a quadric of revolution and *the fixed plane* $x = 0$ *is a meridian plane.* This may be regarded as a limiting case of the general one, and consequently it is easy to understand from geometrical considerations what the law of distribution of the facettes f' becomes. We know from geometrical considerations that the focal conic degenerates, *as an envelope,* into a pair of points, the two foci of the meridian conic, or *principal foci.* The enveloping cone must break up correspondingly into two pencils of planes whose axes are the two lines projecting the principal focal points from the center of f. We can readily confirm this result with our formulas. It will be sufficient to consider only one of the two cases, for example that of the central quadric S_0 with $a = b$. The formulas (36) become in this case

$$A = \frac{a(a+1)}{c}, \qquad B = \frac{a(a+1)}{c}v, \qquad C = \frac{a(a+1)}{c}v^2,$$

so that

$$\Theta = \pm\sqrt{\frac{a(a+1)}{c}}(\lambda + v).^*$$

The formulas (38) give

$$X' \equiv (z_0^2 + a^2 v^2)\lambda - a^2 u^2 v \mp a u^2 z_0 \sqrt{\frac{a(a+1)}{c}}(\lambda + v),$$

$$Y' \equiv - a^2 uv\lambda + (z_0^2 + a^2 u^2) u \mp auvz_0 \sqrt{\frac{a(a+1)}{c}}(\lambda + v),$$

$$Z' \equiv auz_0(\lambda + v) \pm uz_0^2 \sqrt{\frac{a(a+1)}{c}}(\lambda + v),$$

and one sees that between these values subsists the identity

$$uX' + vY' + z_0 Z' = \pm\frac{c}{a}\sqrt{\frac{a(a+1)}{c}}Z'.$$

It results that the planes of the facettes f', whose equations are

$$(x - u)X' + (y - v)Y' + (z - z_0)Z' = 0,$$

all pass through one or the other of the points

$$x = 0, \qquad y = 0, \qquad z = \pm\frac{c}{a}\sqrt{\frac{a(a+1)}{c}},$$

* If it is desired that the results be real, one must take $c > 0$, $a > 0$, or $c > 0$, $a < -1$. The first case is that of the hyperboloid of revolution of two sheets, the second the prolate ellipsoid.

which are precisely the principal foci. Similar results can be established for the paraboloids, in which case, however, it must be realized that there is only one proper principal focus. Hence we have the result:

When in the solution of problem (A) *one takes a quadric of revolution with fixed plane for a meridian plane, the* ∞^1 *facettes* f', *associated with a facette* f *of the quadric, have their planes distributed in a pencil whose axis is a line through the center of* f *projecting one or the other of the two principal foci.*

12. The transformed surfaces S'

The researches set forth above have demonstrated that all the solutions of problem (A) are obtained on assuming that S_0 is a quadric which has the fixed plane for a principal plane and by associating with each of its facettes f a simple infinity of facettes f', according to the geometric law described, with reference to the focal conic in the general case, and the two principal foci in the special case. If now the quadric S_0 assumes, by flexure, any form whatever S, and each facette f together with its associated ∞^1 facettes f' are carried along in invariable relation, it will always be true that the ∞^3 facettes f' will be distributed on ∞^1 surfaces S', transforms of S. Each S' and the primitive S are the focal surfaces of the congruence of lines joining corresponding points on these two surfaces. We will find that this is always a *W-congruence*, that is on S and S' conjugate systems correspond (or, what is the same thing, asymptotic lines correspond). For this it will be sufficient to prove* that one of the focal surfaces, for instance S', admits an infinitesimal deformation in which each of its points (x', y', z') is displaced parallel to the normal (X, Y, Z) to S at the corresponding point. If ρ denotes the unknown amplitude of this displacement, we must verify that the following three conditions are satisfied:

$$S\frac{\partial(\rho X)}{\partial u}\frac{\partial x'}{\partial u} = 0, \qquad S\frac{\partial(\rho X)}{\partial v}\frac{\partial x'}{\partial v} = 0,$$

$$S\left(\frac{\partial(\rho X)}{\partial u}\frac{\partial x'}{\partial v} + \frac{\partial(\rho X)}{\partial v}\frac{\partial x'}{\partial u}\right) = 0.$$

Developing these expressions by means of (10), (11), and (A), § 2, we find that these conditions are reducible to the following two:

$$(39)\qquad \begin{aligned} \frac{\partial \log \rho}{\partial u} &= \frac{p(ur - \lambda s)}{u(1 + p^2 + q^2)} - \Delta'\Theta, \\ \frac{\partial \log \rho}{\partial v} &= \frac{p(us - \lambda t)}{u(1 + p^2 + q^2)} - \Delta''\Theta. \end{aligned}$$

* *Lezioni*, vol. 2, § 242.

By means of the calculations carried out in § 3 we can show that the conditions of integrability of (39) are satisfied, and consequently our assertion is proved.

We shall not carry on further any study of the transformations of deforms of quadrics from this point of view, but we will record that in the general case *all the transforms S' are applicable to one another and to the primitive surface S.* We will add that also in the special case when S_0 is a quadric of rotation with the fixed plane for a meridian plane it turns out that all of the transforms S' are applicable to one another and to a fixed quadric. However, the latter does not in this case coincide with the primitive quadric, but is a quadric (imaginary) of Darboux tangent at one point to the imaginary circle at infinity. The W-congruence which is obtained in this case, with its two focal surfaces applicable to two different quadrics, was discussed from an entirely different point of view in 1906 in my memoir inserted in volume 22 of the Rendiconti del Circolo matematico di Palermo. Thus it is seen that this congruence and those which arise from singular transformations B_k are coördinated by the geometrical principle set forth in problem (A) whose solution has been effected in this memoir.

PISA, ITALY

TYPES OF (2, 2) POINT CORRESPONDENCES BETWEEN TWO PLANES*

BY

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1. **Introduction.** The purpose of this paper is to obtain a classification of the possible $(2, 2)$ point correspondences between two planes, and to describe the important features of each type. Consider two equations each algebraic in two sets of homogeneous coördinates $x_1, x_2, x_3; x_1', x_2', x_3'$. When x_1, x_2, x_3 are given, the equations are to represent two curves in the (x') plane whose two variable intersections, that is, those depending on the parameters x_1, x_2, x_3, are the images of the point $(x) \equiv (x_1, x_2, x_3)$. Similarly if x_1', x_2', x_3' are given x_1, x_2, x_3 have two sets of values. By considering the possible forms of these equations, eleven independent types are obtained. Let P_1 have the images P_1', P_2'; the images of P_1' are P_1 and P_2; the images of P_2' are P_1 and $\bar{P}_2$. Two distinct cases appear: When P_2 and $\bar{P}_2$ are distinct except for restricted positions of P_1, the correspondence will be said to belong to the general case. In this category belong Types I to V. If P_2 is identical with $\bar{P}_2$ for every position of P_1, the correspondence will be called a compound involution. In this category belong Types VI to XI which are developed independently of Types I–V.

Finally it is shown that any $(2, 2)$ point correspondence between two planes is birationally equivalent to one of the types here enumerated.

Only very special cases of $(2, 2)$ correspondences have heretofore been studied, and practically none by the methods here employed.†

2. **The general case.** If the two images of a point which is on the plane (x) coincide, the point is on the curve of branch points. This curve will be designated by the symbol $L(x)$. The locus of the corresponding coincidences is a curve $K'(x')$ which is in $(1, 1)$ correspondence with $L(x)$. Similarly there is a curve of branch-points $L'(x')$, and a curve of coincidences

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† See Pascal's (German) *Repertorium der höheren Mathematik*, 2d edition, vol. 2, p. 371. All the results previously obtained appear as particular cases in our classification. No use is made of any of the papers there cited. See also R. Baldus, *Zur Theorie der gegenseitig mehrdeutigen algebraischen Ebenentransformationen*, Mathematische Annalen, vol. 72 (1912), pp. 1—36.

$K(x)$. The image of $L(x)$ is $K'(x')$ counted twice. The image of $K'(x')$ is $L(x)$ and a residual curve $R(x)$. We now require and shall frequently have occasion to make use of the following lemma.

LEMMA. *Let a point P in (x) describe a curve C. The necessary and sufficient condition that its images P_1', P_2', describe distinct curves is that C touches L at every common point.* This follows immediately from the fact that P_1' and P_2' cannot interchange. Applying the lemma to the curves $K(x)$ and $K'(x')$ we have the theorem.

THEOREM. *The curves $L(x)$, $K(x)$ and $L'(x')$, $K'(x')$ are tangent to each other at their common points.*

If a line c_1 in (x) meets $K(x)$ in k points its image in (x') is a curve c' tangent to $L'(x')$ at k points. If c' has d variable double points, its image in (x) is c_1 counted twice and a curve meeting c_1 in k points corresponding to the k contacts of c' with $L'(x')$ and in $2d$ points corresponding to the d variable double points of c'.

NON-INVOLUTORIAL TYPES

3. **Types of correspondences.** An algebraic correspondence between the points of the planes (x) and (x') may be expressed by two algebraic equations of the form

$$\sum a_i(x')u_i(x) = 0, \tag{1}$$

$$\sum b_i(x')v_i(x) = 0, \tag{2}$$

where $a_i(x') = 0$, $b_i(x') = 0$ belong to linear systems of curves in the (x') plane, and similarly for $u_i(x) = 0$, $v_i(x) = 0$ in the (x) plane. The equations (1) and (2) define in each plane two algebraic systems of curves, the coördinates of a point in the other plane being the parameters. In each plane any curve of one system meets a curve of the other system in two variable points.

The following five cases give independent types but special cases of one type may sometimes be included in one or more of the remaining types.

No.	$a_i(x') = 0$	$u_i(x) = 0$	$b_i(x') = 0$	$v_i(x) = 0$
I.	conic	line	line	conic
II.	conic	conic	line	line
III.	conic	line	conic	conic
IV.	conic	conic	conic	conic
V.	c_n	c_n	line pencil	line pencil

where c_n is a curve of order n with the basis point of the corresponding line pencil $(n-2)$-fold.

We proceed to discuss briefly each of these cases.

4. **Type I.** Image of a line. We may write the defining equations in the

form

$$(1') \qquad \sum b_i(x)x_i' \equiv \sum a_{ik}'(x')x_i \cdot x_k = 0,$$

$$(2') \qquad \sum b_i'(x')x_i \equiv \sum a_{ik}(x)x_i' \cdot x_k' = 0,$$

wherein a_{ik}, a_{ik}' are linear, b_i, b_i' quadratic in the respective variables.

The image of a line

$$(3) \qquad \sum k_i x_i = 0$$

is the quintic

$$(4) \qquad \sum a_{ik}'(x')X_i'X_k' = 0,$$

wherein X_i' is the cofactor of x_i' in the determinant

$$\begin{vmatrix} x_1' & x_2' & x_3' \\ b_1' & b_2' & b_3' \\ k_1 & k_2 & k_3 \end{vmatrix}.$$

The quintic (4) is of genus 2, having double points at the four points given by

$$(5) \qquad \frac{b_1'}{k_1} = \frac{b_2'}{k_2} = \frac{b_3'}{k_3}.$$

5. **Curves of branch points and of coincidences.** The two images of a point of (x') coincide if the line (2′) touches the conic (1′). The curve of branch points in (x') is therefore the sextic of genus 10

$$L'(x') \equiv \begin{vmatrix} a_{11}' & a_{12}' & a_{13}' & b_1' \\ a_{21}' & a_{22}' & a_{23}' & b_2' \\ a_{31}' & a_{32}' & a_{33}' & b_3' \\ b_1' & b_2' & b_3' & 0 \end{vmatrix} = 0.$$

The curve of coincidences $K'(x')$ counted twice is the image of $L(x)$, which is similar to $L'(x')$. It is of order 15 and genus 10, and has 45 contacts with $L'(x')$.

6. **Locus of double points in the images of a pencil of lines.** The images of the point $(0, 0, 1)$ in (x) lie on the conic $b_3' = 0$. From (5) this conic also passes through the four double points of the quintic corresponding to any line through $(0, 0, 1)$. Hence the theorem:

THEOREM. *The two images of a point and the double points of the quintic images of the pencil of lines through the point lie on a conic.*

7. **Successive images of a line.** It has just been shown that the image of a line c_1 in (x) is a quintic c_5' having four double points and 15 contacts with L_6' which correspond to the 15 intersections of c_1 with K_{15}. The image of c_5' is therefore c_1 counted twice and a residual c_{23} of genus 2 passing through the 15 intersections of c_1 with K_{15} and through 8 other points on c_1, images of the 4 double points of c_5'. The images of two lines c_1, $\bar{c}_1$ are two quintics

c_5', $\bar{c}_5'$ which meet in 25 points. Two of these points correspond to the common point of c_1, $\bar{c}_1$. The remaining 23 have for images the 23 intersections of c_1 with $\bar{c}_{23}$ and $\bar{c}_1$ with c_{23}. Since the curve c_{23} has a composite image it has 69 contacts with L_6. The 75 intersections of c_5' with K_{15}' are the images of these 69 points and of the 6 intersections of c_1 with L_6. The image of c_{23} is c_5' and a residual c_{110}' passing through the 4 double points of c_5' and the 15 contacts of c_5' with L_6'. The curve c_{110}' has 330 contacts with L_6' which correspond to the 330 points in which c_{23} meets K_{15} apart from the 15 which lie on c_1. The curves c_5', c_{110}' meet in 550 points, namely 8 for the 4 double points of c_5', 15 for the points of contact of c_5' with L_6', 69 on K_{15}', images of the 69 contacts of c_{23} and L_6, and finally 458 images of the 229 double points of c_{23}. By carrying the process one step further we can see the general law. The image of c_{110}' is c_{23} and a residual c_{527} which meet in 12121 points, made up as follows: the 23 intersections of c_1, c_{23}; 330 points of c_{23} on K_{15} but not on c_1; 5884 pairs of points, images of the double points of c_{110}'. The curve c_{527} does not pass through the double points of c_{23}, although c_{110}' passes through the double points of c_5'. The reason of this is that both images of a point of c_1 are on c_5', while c_{23} and c_{527} are in $(1, 1)$ correspondence.

8. **Case of conics in (x') with one basis point.** Let the basis point be taken as $(1, 0, 0) \equiv A'$. The image of c_1 is c_5' with one double point at A' and three variable double points. The curve L_6' has a double point at A' and is of genus 9. For points in the neighborhood of A' we have

$$b_1 = 0, \qquad a_{12} x_2' + a_{13} x_3' = 0,$$

hence to each direction through A' correspond 2 points on the conic $b_1 = 0$ which is the image of A'. The image of c_5' is c_1^2, the conic b_1 counted twice and a residual c_{19} of genus 2. The image of L_6' is b_1 and K_{13} of genus 9. The curve c_{19} meets c_1 in 13 points on K_{13} and in the 6 points—images of the three variable double points of c_5'. The conic b_1 touches L_6 in 6 points; hence K_{15}' has A' sixfold and 66 double points. The image of K_{15}' is $L_{63}(b_1)^6$ and a residual C_{57} touching L_6 in the 39 points of contact with K_{13} and in the 132 point-images of the 66 double points of K_{15}'. The image of b_1 is the point A' and a rational C_{10}' having A' sixfold. The curves L_6', c_{10}', meet in 60 points, namely 12 at A_1, two contacts, images of the contacts of b_1 and K_{13}, since the two images on b_1 of a tangent to L_6' coincide, and in 22 other contacts, images of the 22 remaining intersections of b_1 and K_{13}.

A line in (x') through A' has for image the conic b_1 and a cubic curve of genus 1. Two lines of the pencil A' have no other point in common, hence the nine points of intersection of their image cubics must be accounted for. The image of c_3 is a curve of order 15 having A' for a sixfold point; but c_1' appears twice as a component, hence the residual is a c_{13}', having A for a

fourfold point. The curve c'_{13} meets c'_1 in 9 points besides A; they are at the nine points in which c'_1 meets K'_{15} besides A, which was seen to be a sixfold point on K'_{15}.

Now consider two lines c'_1, $\bar{c}'_1$; their images c_3, $\bar{c}_3$ meet in 9 points, whose images in (x') lie 9 on c'_1, $\bar{c}'_{13}$ and 9 on $\bar{c}'_1$, c'_{13}. Two of the intersections of c_3 with b_1 are images of the direction of c'_1 through A.

A curve of order n in (x) meets b_1 in $2n$ points; if these points are all independent, so that no two of them correspond to the same direction through A, the image curve in (x') will have a $2n$-fold point at A. But if the curve goes through a pair of conjugate points both points indicate but a single tangent to the image curve at A. A third alternative is illustrated by the images of an arbitrary straight line. Let c_1 meet b_1 in P_1, P_2. The images of P_1, P_2 are the two tangents t_1, t_2 to c'_5 at A, each of which has another image $\bar{P}_1$, $\bar{P}_2$ on b_1. The residual image of c'_5 is c_{19}, which meets b_1 in 38 points. But two of these points are $\bar{P}_1$, $\bar{P}_2$. The two images of $\bar{P}_1$ are t_1 and a point in (x') on the residual image of c_{19}; similarly for $\bar{P}_2$. Hence the residual image of c_{19} has A for a multiple point of order 36. This alternative is possible only when the image of the curve under consideration is composite.

9. **Case of conics in (x) and (x') with one basis point.** Let the basis points be A and A' each $(1, 0, 0)$ in (x) and (x'). The image of A is $b'_1 = 0$ through A', and of A' is $b_1 = 0$ through A. Hence we have the following theorem.

THEOREM. *If the conics in each plane have a basis point, each basis point lies on the image conic of the other.*

The details of this case are in other respects similar to the preceding case.

10. **Case of conics in (x') having two basis points.** Let the basis points be $A' \equiv (1, 0, 0)$ and $B' \equiv (0, 1, 0)$. For points on the line $A'B'$ we have $b_1 x'_1 + b_2 x'_2 = 0$, $a_{12} = 0$. Hence to a point on $A'B'$ correspond two points on the line $a_{12} = 0$. The image of the line $A'B'$ is b_1, b_2, and a_{12}.

The image of a_{12} is $A'B'$ taken twice, and a rational c'_3 not passing through A' or B'. The line a_{12} is a tri-tangent to L_6, hence $A'B'$ meets c'_3 in three points on K'_{15}. Since K'_{15} has A' and B' for sixfold points, $A'B'$ has no other points on K'_{15}.

The line $A'B'$ meets L'_6 in two points besides A', B'. The images of these points are the points of contact of the conics of the pencil b_1, b_2 which touch a_{12}.

The cases of two basis points in one plane, together with one or two in the other plane present no difficulties.

11. **Three basis points in (x'), none in (x).** Since they can not all be collinear, we may use the triangle having the basis points for vertices as triangle of quadratic inversion and reduce the conics of the system through them to straight lines, and the straight lines expressed in the other equation to conics,

thus reducing the defining equations to a particular case of Type II. Similar transformations can be made when the lines have a basis point. They can be transformed into a pencil of conics whether the coefficients are linear or quadratic. In either case the curves defined by the same equation in the other plane also belong to a pencil.

12. **Other particular forms of Type I.** Suppose $x_1 = 0, x_1' = 0$ satisfy both equations. The image of c_1 has $x_1' = 0$ as a fixed component, the variable component being a quartic of genus 1. The curve L_6 has $x_1^2 = 0$ as a component, the residual curve being a quartic of genus 1.

If $x_2 = 0, x_2' = 0$ also satisfy both equations, there are three basis points common to the conics in each plane. The proper image of a straight line is now a rational cubic. The curve L_2 is a conic and K_3 a rational cubic.* If finally both equations are also satisfied by $x_3 = 0, x_3' = 0$ they may be written in the form

$$x_1' x_2 x_3 + x_2' x_3 x_1 + x_3' x_1 x_2 = 0,$$

$$a_1 x_1 x_2' x_3' + a_2 x_2 x_3' x_1' + a_3 x_3 x_1' x_2' = 0.$$

The fundamental points in each plane are

$$(1, 0, 0), \qquad (0, 1, 0), \qquad (0, 0, 1),$$

and the associated fundamental conics are

$$x_2 x_3 = 0, \qquad x_3 x_1 = 0, \qquad x_1 x_2 = 0.$$

Each fundamental point is a double point on a fundamental conic. The equations define two collineations of the form $x_i = k_i x_i'$ which can be rationally separated.

Both L and K vanish identically.

13. **Type II.** A straight line in either plane has for image a quartic with one variable double point. The curve L is of order 6 and genus 10; K is of order 12. These two curves have 36 points of contact. For every basis point of the conics in either plane L_6 has a double point, and in the other plane there is a fundamental line. The extreme particularization

$$b_1(x) x_1' + b_2(x) x_2' = 0, \qquad b_1'(x') x_1 + b_2'(x') x_2 = 0$$

having five basis points in each plane, with four fundamental conics belonging to a pencil, and a fundamental line, is a direct generalization of the Seydewitz† method of defining a birational quadratic inversion. All the properties of

* This is one of the cases treated by Marletta, *Rendiconti del Circolo Matematico di Palermo*, vol. 17 (1903), pp. 173–184, 371–385. The lines $x_1 = 0, x_1' = 0$ are illustrations of the curves D defined by Baldus, l. c. The curves D appear only in subcases of more general types in our classification.

† *Archiv der Mathematik und Physik*, vol. 7 (1846), pp. 113–148.

this transformation can be derived readily by the methods given in the present paper.

14. **Particular cases of Type II.** If both equations are satisfied by $x_1 = 0$, $x_1' = 0$ the image of a line is a cubic curve of genus 1. The curve L is a quartic of genus 3 and K a sextic. If $x_2 = 0$, $x_2' = 0$ also satisfy both equations the image of a line is a conic with no basis points. The curves L and K are both conics having double contact with each other.*

15. **Type III.** Consider the conics in (x) through $B \equiv (0, 1, 0)$ and $C \equiv (0, 0, 1)$

$$ax_1^2 + bx_1x_2 + cx_1x_3 + dx_2x_3 = 0,$$

$$px_1^2 + qx_1x_2 + rx_1x_3 + sx_2x_3 = 0,$$

where a, b, c, d are quadratic and p, q, r, s linear in (x'). Proceeding as in Type I the image of c_1' is c_6, a sextic with triple points at B and C and two variable double points. L is of order 8 and genus 9 having fourfold points at B and C. To find the image of a point near B we have

$$bx_1 + dx_3 = 0, \qquad qx_1 + sx_3 = 0.$$

Hence to a direction through B correspond 2 points on the cubic $bs - dq = 0$, which is the image of B. Similarly for C. Corresponding to any point on BC are the two points $B'\,C'$ given by $d = s = 0$. To find the image of a line $c_1 \equiv \Sigma k_i x_i = 0$, replace the first of the defining equations by

$$(as - pd)x_1 + (bs - qd)x_2 + (cs - rd)x_3 = 0.$$

Hence proceeding as in Type I, the image of a line is a curve of order 7 with double points at the 9 points given by

$$\frac{as - pd}{k_1} = \frac{bs - qd}{k_2} = \frac{cs - rd}{k_3},$$

of which 2 are at B' and C' and 7 variable. But $s = 0$ is a fixed component so that the image of c_1 is c_6' having 7 double points and passing through B and C. Similarly L' is of order 6 because the curve of order 8 obtained contains $s = 0$ twice as an extraneous factor. The curve K is of order 18, with B and C 9-fold points. The curve K' is of order 12.

16. **Type IV.** If in the equations used for Type III the coefficients of the (x)'s are all quadratic in the (x')'s and have no terms in $x_2'^2$ or $x_3'^2$ there are 2 basis points $B' \equiv (0, 1, 0)$ and $C'' \equiv (0, 0, 1)$. Proceeding as in Type III to replace one of the equations by

$$(as - pd)x_1 + (bs - qd)x_2 + (cs - rd)x_3 = 0,$$

the image of a line

$$c_1 \equiv \sum k_i x_i = 0$$

* Both cases were discussed by Marletta, l. c., § 12.

is c_8' with B' and C'' fourfold points, 6 variable double points and passing simply through E' and F', the residual intersections of $d = 0$, $s = 0$. The curve L' is of order 8, with B' and C'' fourfold points. For points near B, we have

$$bx_1 + dx_3 = 0, \qquad qx_1 + sx_3 = 0.$$

Hence to each direction through B correspond two points on the quartic $bs - dq = 0$, which has double points at B' and C', simple points at E' and F' and is the image of B. The image of a point on $B'C''$ is E and F. The image of a general curve of order 8 in (x') is of order 64, with B and C for 32-fold points, and E and F 8-fold points. In the case of L_8' the quartic images of B' and C'' are to be taken off each 4 times, leaving for K^2 a curve of order 32, having B and C 16-fold. Hence K is of order 16, with B and C each 8-fold. In the case of c_8' we must also take off BC (the image of E and F) twice, and also c_1'. The residual is of order 28 with B and C 14-fold points.

17. **Type V.** The equations (1), (2) have the forms

$$\sum b_i b_i' = 0, \qquad x_1 x_1' + x_2' x_2 = 0,$$

wherein

$$b_i = x_3^2 \phi_{n-2,\, i}(x_1, x_2) + x_3 \phi_{n-1,\, i}(x_1, x_2) + \phi_{n,\, i}(x_1, x_2) = 0$$

and b_i' is similar, but of order n'.

The point $(0, 0, 1)$ in each plane is the only fundamental point. Its image is of order $n + n' - 2$ having $(0, 0, 1)$ $(n + n' - 4)$-fold. The image of a line is a curve of order $n + n'$ having $(0, 0, 1)$ $(n + n' - 2)$-fold. The curves of branch points are of order $2n + 2n' - 2$, having $(0, 0, 1)$ $(2n + 2n' - 6)$-fold. The curves of coincidences are of order $4n + 4n' - 6$, having $(0, 0, 1)$ $(4n + 4n' - 10)$-fold.

Compound involutions

18. **Forms of the equations.** In the compound involutions, a point P_1 has P_1', P_2' for images; the images in (x) of P_1' are P_1 and P_2; those of P_2' are the same points P_1 and P_2. Hence, if P_1 or P_2 is given, the other is uniquely determined, that is, in each plane the two images of a point in the other belong to a simple involution. This fact enables us to map the plane (x) on a double plane (y) by means of a $(1, 2)$ transformation, and similarly the plane (x') on a double plane (y'). The planes (y) and (y') must be birationally equivalent. An involution can always be defined by a net of curves, and mapped on a double plane (y) by equations of the form

$$\frac{\phi_1(x)}{y_1} = \frac{\phi_2(x)}{y_2} = \frac{\phi_3(x)}{y_3},$$

where $\phi_i = 0$ define a net of curves with two intersections that are functions of y_1, y_2, y_3. The second transformation is

$$y'_i = \psi_i(y),$$

a Cremona transformation, and the third is

$$\frac{\phi'_1(x')}{y'_1} = \frac{\phi'_2(x')}{y'_2} = \frac{\phi'_3(x')}{y'_3},$$

wherein the curves define a net in (x') with two intersections that are functions of y'_1, y'_2, y'_3.

By eliminating y_1, y_2, y_3; y'_1, y'_2, y'_3 from these equations we have the following theorem.

THEOREM. *The necessary and sufficient condition that a* $(2, 2)$ *transformation is a compound involution is that the equation which define it may be reduced to the form*

$$\frac{F_1(x)}{F'_1(x')} = \frac{F_2(x)}{F'_2(x')} = \frac{F_3(x)}{F'_3(x')},$$

in which any two curves represented by these equations and lying in one plane meet in two variable points. In geometric form, the theorem states that the sufficient condition that a $(2, 2)$ transformation is a compound involution is that the defining curves in one of the planes (and hence also in the other) compose a net.

Compound involutions appear as particular cases of each of the types already mentioned, but others exist which cannot be thus expressed.

19. **Properties of compound involutions.** Let c be any curve in (x). As in the preceding cases, the image of c is a curve c', touching L' at every common point; but in the present case, the image of c' in (x) consists of c and of a residual curve $\bar{c}$, each counted twice. The curve $\bar{c}$ is not a contact curve of L, and c' is the complete image of $\bar{c}$.

When P describes L, its images $P'_1 \equiv P'_2$ describe K', and the residual image of P'_1 also lies on L. Hence the image of L is K' counted four times, and the complete image of K' is L. Hence L, K' are not in $(1, 1)$ correspondence. Since the image of K is not composite, K is not a contact curve of L.

The curves K, K' are the jacobians of the nets in their respective planes.

In the following classification, two transformations are regarded as equivalent when their two component $(1, 2)$ transformations are respectively equivalent. For simplicity it is here assumed that linear relations between (y) and (y') exist, such that the fundamental elements of the two $(1, 2)$ transformations are distinct.

20. **The $(1, 2)$ transformations.** The $(1, 2)$ plane transformations have

been extensively studied.* They are of three types. The first is obtained by the intersections of a line of a plane field with an associated conic of a net, or of the cubics of a net through seven fixed points. It will be called the Geiser type. The second is given by the intersection of a line of a pencil 0 with a curve of order n of a net having 0 for $(n-2)$-fold point. This will be called the Jonquières type. The third is found by the variable intersections of a cubic belonging to a pencil, with an associated sextic of a pencil having eight of the basis points of the cubics for double points. It will be called the Bertini type. By combining these various types we obtain the following Types of $(2, 2)$ compound involutions.

VI. Geiser, Geiser.
VII. Geiser, Jonquières.
VIII. Geiser, Bertini.
IX. Jonquières, Jonquières.
X. Jonquières, Bertini.
XI. Bertini, Bertini.

Before discussing these types it will be convenient to state the principal properties of the three component simple involutions.

The Geiser type. The image of a line $c_1(y)$ is a cubic with seven simple basis points A_i. The curve K is a sextic, having each point A_i double. The curve $L(y)$ is a quartic of genus 3. There are no fundamental elements in (y). A line $c_1(x)$ goes into a cubic curve with one variable double point and touching $L(y)$ in six points. The image of a point A_i is a bitangent to $L(y)$.

The Jonquières type. A line $c_1(y)$ goes into $c_n(x)$ having $C_x(n-2)$-fold and $4n-6$ simple basis points $B_i(x)$. The point C_y goes into $C_{n-1}(x)$ having $C_x(n-3)$-fold and the points $B_i(x)$ simple. The curve $K(x)$ is of order $2n-2$ having $C_x(2n-4)$-fold and the points B_i simple. A line $c_1(x)$ goes into $C_n(y)$ having $C_y(n-1)$-fold. The point C_x goes into $C_{n-2}(y)$ having $C_y(n-3)$-fold. Each point B_i goes into a line through C_y. The curve L_y is of order $2n-2$ having $C_y(2n-4)$-fold.

The Bertini type. The line $c_1(y)$ goes into $c_6(x)$, having 8 double points A_i and 2 simple basis points D_i. The curve $K(x)$ is of order nine having the points A_i triple. The curve $L(y)$ is a sextic with two consecutive triple points at a point $E(y)$, the tangent being a line γ. The image of $E(y)$ is a cubic through the points A_i and D_i. The image of a point A_i is a conic touching γ at $E(y)$ and touching $L(y)$ in three other points. The image of a point D_i is the line $D_y E_y \equiv \gamma$. The line $c_1(x)$ goes into a sextic with two consecutive triple points at $E(y)$, touching $L(y)$ in nine points and having four variable double points.

* See Pascal's *Repertorium*, l. c., pp. 366–370, for the principal literature.

21. **Type VI.** Each fundamental point A_i goes into an elliptic cubic of the net in (x'). A line c_1 goes into c'_9 with seven triple points at A'_i, and two variable double points.

K_6 goes into L'_{12} having A'_i for fourfold points. It is of genus 13.

The image of c'_9 is a curve of order 81 in (x), consisting of seven fundamental cubics, images of A'_i, each counted three times, the original c_1 counted twice, and the rational c_8 with triple points at A_i, image of c_1 in the Geiser involution in (x), also counted twice. The line c_1 and the curve c_8 have no restricted position as to L, but c'_9 touches L' in each common point. The curve c_9 is the complete image of c_8.

22. **Type VII.** The fundamental points A_i in (x) go into seven curves of a Jonquières net, of order n. The only fundamental points in (x') are the point $C' = (0, 0, 1)$, $(n-2)$-fold on all curves of order n, and $4n-6$ simple basis points B'_i. The (y) image of the multiple point is a curve of order $n-2$, with an $(n-3)$-fold point C_y, and its image in (x) is of order $3(n-2)$, having seven points of order $n-2$ at A_i, and two points C_1, C_2, each of multiplicity $n-3$. The (y) images of the simple points are $4n-6$ lines of the pencil C_y; the (x) images of these lines are $4n-6$ cubics of the pencil through the seven basis points A_i and C_1, C_2.

A line $c_1(x)$ goes into a nodal c_3 in (y), which goes into c'_{3n}, having C' of multiplicity $3(n-2)$, and having $4n-6$ threefold points at B'_i. A line c'_1 goes into $c_n(y)$ with C_y for $(n-1)$-fold point; its image in (x) is c_{3n} with n-fold points at A_i and two $(n-1)$-fold points C_1, C_2. The curve K_6 goes into $c_4(y)$, whose image L' in (x') is of order $4n$, having C' as $4(n-2)$-fold point, and $4n-6$ points B'_1, each of multiplicity 4. The curve $K'_{2(n-1)}$ goes into $L_{2(n-1)}(y)$ with C_y of multiplicity $2(n-2)$ and this goes into $L_{6(n-1)}$ with A_i each of multiplicity $2(n-1)$ and two points C_1, C_2, each of multiplicity $2(n-2)$.

23. **Type VIII.** The fundamental points A_i go into seven curves of order 6 belonging to the net in (x'). The points E'_i go into eight sextics having double points at A_i. The points D_i go into a cubic of the net. A line c_1 goes into c'_{18} with sixfold points at E'_i, two triple points at D'_1, D'_2, and two variable double points. The line c'_1 goes into c_{18} with sixfold points at A_i, two consecutive triple points at F_1, F_2, and eight variable double points.

The curve L'_{24} has eightfold points at E'_i, and fourfold points at D'_1, D'_2, while L_{18} has consecutive triple points at F_1, F_2, and sixfold points at A_i.

The residual image of c'_{18} is c'_{17}, image of c_1 in the Cremona involution of Bertini type.

24. **Type IX.** A line c_1 goes into $c_n(y)$ with C_y as $(n-1)$-fold point; this goes into $c'_{nn'}$, having C' as point of order $n(n'-2)$, two points of multiplicity $n-1$, and having $4n'-6$ points, each of multiplicity n. The curve

K_{2n-2} has C of multiplicity $2n-4$, and $4n-6$ simple points B_i, hence $L'_{2n(n-1)}$ has C' of order $2(n-1)(n'-2)$, $4n'-6$ points B_i each of order $2(n-1)$, and two points of order $2n-4$. By interchanging n and n', we obtain K' and L.

25. **Type X.** The points E_i go into curves of order $2n$, having C' as $2(n-2)$-fold point, $4n-6$ points B_i for double points. The point C' goes into a curve of order $6(n-2)$ having E_i for $2(n-2)$-fold points, two $(n-3)$-fold points, and two $(n-2)$-fold points D_1, D_2. A line c_1 goes into c'_{6n}, having C' as $6(n-2)$-fold point, B'_i for sixfold points, two consecutive triple points at F'_1, F'_2, and eight variable double points.

A line c'_1 goes into c_{6n}, with two points C_1, C_2 of multiplicity $n-1$, the points E_i of multiplicity $2n$, and D_1, D_2 n-fold. The curve L is of order $12(n-1)$, has E_i for points of multiplicity $4(n-1)$, D_1, D_2 of multiplicity $2(n-1)$, C_1, C_2 of multiplicity $2(n-2)$. The curve L' is of order $6n$, has consecutive triple points at F'_1, F'_2 and C' for point of multiplicity $6(n-2)$, B'_i of multiplicity 6.

26. **Type XI.** The points E_i go into curves of order 12 having E'_i for fourfold points, and D'_1, D'_2 for double points. The line c_1 goes into c'_{36} having 12-fold points at E'_i, sixfold points at D'_1, D'_2, two consecutive triple points at F'_1, F'_2 and eight variable double points.

The curve L'_{36} has eight 12-fold points at E'_i, sixfold points at D'_1, D'_2, and two consecutive triple points at F'_1, F'_2.

27. **Relations between the two lists.** If the equations (1) and (2) each contain but two terms the resulting particular forms of Types I–V can be expressed as follows:

(1) $\quad u_1(x)x'_1 + u_2(x)x'_2 = 0, \qquad u'_1(x')x_1 + u'_2(x')x_2 = 0,$

(2) $\quad u_1(x)u'_1(x') + u_2(x)u'_2(x') = 0, \qquad x_1x'_1 + x_2x'_2 = 0,$

(3) $\quad v_1(x)x'_1 + v_2(x)x'_2 = 0, \qquad v_3(x)u'_1(x') + v_4(x)u'_2(x') = 0,$

(4) $\quad v_1(x)v'_1(x') + v_2(x)v'_2(x') = 0, \qquad v_3(x)v'_3(x') + v_4(x)v'_4(x') = 0,$

(5) $\quad w_n(x)w'_n(x') + \bar{w}_n(x)\bar{w}'_n(x') = 0, \qquad x_1x'_1 + x_2x'_2 = 0,$

in which $u_i = 0$ is a general conic, $v_i = 0$ a conic through two fixed points, and $w_n = 0$ a curve of order n having $(0, 0, 1)$ for a point of multiplicity $n-2$.

All these forms are included in (5), which is a particular form of Type IX.

28. **Fixed loci.** Suppose a curve c of one system meets a curve k of another system in s variable points, of which $s-2$ always lie on a fixed curve, leaving but two variable intersections. It can be shown that such cases can always be reduced to one or another of those previously considered, even when such fixed loci exist in both planes.

29. **Proof of non-existence of other types.** It remains to be proved that any $(2,2)$ point correspondence between two planes can be reduced by birational transformation to one of the eleven types already obtained. The curves defined by (1) and (2) each belong to linear systems, any curve of one system having two variable intersections with the curves of the other system in the same plane. In the determination of pairs of systems of this kind in a plane (x), it should be remembered, that by a birational transformation of the plane, and by taking as parameters functions which define the curves of a suitably chosen cremona net in a plane (y) we can use any such pair to set up a $(1,2)$ correspondence between the planes (x) and (y), which is either a Geiser or Jonquières or Bertini correspondence, expressed in the normal form. We need therefore consider only such pairs of linear systems, as can be used to define one of these three correspondences. We therefore have (1) lines and conics, (2) conics and conics each with the same two basis points, (3) line pencil vertex O and curves of order n, having O $(n-2)$-fold, (4) the image in (x) of a pair of systems in (y) having one variable intersection, namely either a Cremona net or the lines of the plane (y), or a line pencil vertex O and curves of order n having O $(n-1)$-fold.

The first three cases give the Types I to V, the fourth gives the Types VI to XI. The classification is therefore complete.

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